Math 125, Section E, Spring 2011,Solutions to Midterm II

1. (a) Use integration by parts

$$u = (\ln x)^2 \qquad\qquad dv = dx$$

$$du = \frac{2\ln x}{x}dx \qquad\qquad v = x$$

Then

$$\int_1^5 (\ln x)^2 dx = x(\ln x)^2\Big|_1^5 - 2\int_1^5 \ln x dx$$

and integration by parts for a second time

$$u = (\ln x) \qquad\qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad\qquad v = x$$

then the integral becomes

$$x(\ln x)^2\Big|_1^5 - 2x\ln x\Big|_1^5 + 2\int_1^5 dx = 5(\ln 5)^2 - 10\ln 5 + 8 \approx 4.857.$$

(b) Use Simpson's rule with $n = 6$ to estimate the integral in part (a) and compare your answers.

$$S_6 = \frac{4}{18}\left((\ln 1)^2 + 4(\ln\frac{10}{6})^2 + 2(\ln\frac{14}{6})^2 + 4(\ln\frac{18}{6})^2 + 2(\ln\frac{22}{6})^2 + 4(\ln\frac{26}{6})^2 + (\ln 5)^2\right) \approx 4.861.$$

2. Evaluate the following definite integrals.

(a) You can use substitution $u = 4 - x^2$

$$\int_0^1 \frac{x^3}{\sqrt{4-x^2}}dx = \int_4^3 \frac{4-u}{\sqrt{u}}\left(-\frac{1}{2}\right)du = \frac{1}{2}\int_3^4 4u^{-1/2} - u^{1/2}du = 4u^{1/2} - \frac{1}{3}u^{3/2}\Big|_3^4 = \frac{16}{3} - 3\sqrt{3}.$$

You can also solve the problem using inverse trigonometric substitution

$$x = 2\sin\theta \qquad\qquad dx = 2\cos\theta d\theta$$

the integral becomes

$$\int_0^1 \frac{x^3}{\sqrt{4-x^2}}dx = \int_0^{\pi/6} \frac{8\sin^3\theta}{2\cos\theta}2\cos\theta d\theta = 8\int_0^{\pi/6}\sin^3\theta d\theta = 8\int_0^{\pi/6}(1-\cos^2\theta)\sin\theta d\theta$$

now let $u = \cos\theta$

$$= -8\int_1^{\sqrt{3}/2} 1 - u^2 du = -8\left(u - \frac{u^3}{3}\right)\Big|_1^{\sqrt{3}/2} = -3\sqrt{3} + \frac{16}{3}$$

(b) First, substitute $u = e^x$ then complete the square in the denominator,

$$\int_0^{\ln 2} \frac{e^x}{e^{2x} + 6e^x + 10}dx = \int_1^2 \frac{1}{u^2 + 6u + 10}du = \int_1^2 \frac{1}{(u+3)^2 + 1}du$$

then substitute $w = u + 3$:

$$= \int_4^5 \frac{1}{w^2+1}dw = \arctan w\Big|_4^5 = \arctan 5 - \arctan 4.$$

3. For the integral we need partial fractions

$$\frac{x+6}{x^3+3x^2} = \frac{x+6}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

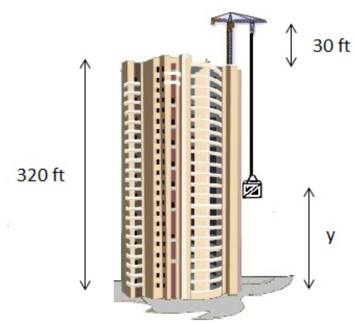
The values are: $A = -1/3$, $B = 2$ and $C = 1/3$. So

$$\int_1^\infty \frac{x+6}{x^3+3x^2}dx = \lim_{t\to\infty} \int_1^t \frac{x+6}{x^3+3x^2}dx = \lim_{t\to\infty} \int_1^t \frac{-1/3}{x} + \frac{2}{x^2} + \frac{1/3}{x+3}dx =$$

$$= \lim_{t\to\infty} \left(-\frac{1}{3}\ln|x| - \frac{2}{x} + \frac{1}{3}\ln|x+3| \Big|_1^t \right) = \lim_{t\to\infty} \left(-\frac{2}{t} + \frac{1}{3}\ln\left(\frac{t+3}{t}\right)\right) - \left(-2 + \frac{1}{3}\ln 4\right) = 2 - \frac{\ln 4}{3}.$$

The integral converges to $2 - \frac{\ln 4}{3}$.

4. A 30 ft crane is lifting a container full of concrete to the top of a 320 ft building. The container itself is 500 lbs and holds 6000 pounds of concrete. Initially, it is full and on the ground. The chain connecting the container to the crane weighs 7 lbs/ft. The crane pulls the chain at a rate of 360 ft/min and concrete leaks out at a rate of 3 lbs/min. Find the work done in lifting this container (and chain) to the top of the building. Hint: First, find the weight of the container+concerete+chain when the container is y feet from the ground.



When the container is at a height of y feet above ground crane is lifting the sum of the weights of

container: 500 lbs

concrete: $6000 - \frac{y}{120}$ lbs

chain: $7(350 - y)$ lbs

So the force on the crane is

$$f(y) = 500 + \left(6000 - \frac{y}{120}\right) + 7(350 - y)$$

and the work done by the crane is

$$W = \int_0^{320} f(y)\ dy = \int_0^{320} 500 + \left(6000 - \frac{y}{120}\right) + 7(350 - y)\ dy$$

$$= \int_0^{320} 8950 - \frac{841}{120}y\ dy = 8950y - \frac{841}{240}y^2\Big|_0^{320} = \frac{7515520}{3}\text{ft.lbs.} \approx 2505173\text{ft.lbs.}$$