Math 125, Section A, Spring 2012, Solutions to Midterm II

1. Evaluate the following integrals.

   (a)
$$\int \frac{x}{\sqrt{3+2x-x^2}} \quad dx = \int \frac{x}{\sqrt{4-(x-1)^2}} \quad dx$$
   using the substitution $x-1 = 2\sin\theta$ we get
$$\int 2\sin\theta + 1 \quad d\theta = -2\cos\theta + \theta + C = -\sqrt{3+2x-x^2} + \sin^{-1}\left(\frac{x-1}{2}\right) + C$$

   (b) Do the substitution $w = \sqrt{x}$ and then do integration by parts on the resulting integral
$$\int e^{\sqrt{x}} \quad dx = \int 2we^w dw = 2we^w - 2e^w + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

2. Evaluate the following integrals.

   (a) Do the substitution $u = \ln x$:
$$\int_1^e \frac{(\ln x)^2}{x} \quad dx = \int_0^1 u^2 du = \frac{1}{3}$$

   (b) The integration techniques is partial fractions:
$$\int_1^\infty \frac{17}{6x^2+13x-5} dx = \lim_{t\to\infty} \int_1^t \frac{3}{3x-1} - \frac{2}{2x+5} dx = \lim_{t\to\infty} \left[\ln|3x-1| - \ln|2x+5|\Big|_1^t\right]$$
$$= \lim_{t\to\infty} \left[\ln\left|\frac{3x-1}{2x+5}\right|_1^t\right] = \lim_{t\to\infty} \left[\ln\left|\frac{3t-1}{2t+5}\right| - \ln\left|\frac{2}{7}\right|\right] = \ln\left|\frac{3}{2}\right| - \ln\left|\frac{2}{7}\right| = \ln\left(\frac{27}{4}\right)$$

3. Use Simpson's rule with $n=6$ to estimate the length of the curve $y = x\sin x$ from $x=0$ to $x=\pi$. Give your answer in exact form (all trig functions should be evaluated) and as a decimal.

   The arclength integral is
$$\int_0^\pi \sqrt{1+(\sin x + x\cos x)^2} dx$$
   So Simpson's Rule with $n=6$ gives
$$S_6 = \frac{\pi}{3\cdot 6}\Bigg[\sqrt{1} + 4\sqrt{1+\left(\sin\frac{\pi}{6} + \frac{\pi}{6}\cos\frac{\pi}{6}\right)^2} + 2\sqrt{1+\left(\sin\frac{2\pi}{6} + \frac{2\pi}{6}\cos\frac{2\pi}{6}\right)^2} + 4\sqrt{1+\left(\sin\frac{3\pi}{6} + \frac{3\pi}{6}\cos\frac{3\pi}{6}\right)^2}$$
$$+2\sqrt{1+\left(\sin\frac{4\pi}{6} + \frac{4\pi}{6}\cos\frac{4\pi}{6}\right)^2} + 4\sqrt{1+\left(\sin\frac{5\pi}{6} + \frac{5\pi}{6}\cos\frac{5\pi}{6}\right)^2} + \sqrt{1+(\sin\pi + \pi\cos\pi)^2}\Bigg]$$
$$= \frac{\pi}{18}\left[1 + 4\sqrt{1+\left(\frac{1}{2} + \frac{\sqrt{3}\pi}{12}\right)^2} + 2\sqrt{1+\left(\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)^2} + 4 + 2\sqrt{1+\left(\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right)^2} + 4\sqrt{1+\left(\frac{1}{2} - \frac{5\sqrt{3}\pi}{12}\right)^2} + \sqrt{1+\pi^2}\right]$$

4. The giants are building a sandcastle for their queen. It is in the shape of a frustrum of a cone (a cone with its top cut off) whose base radius is 10 m, the top radius is 6 m and height is 15 m. Sand of density 1800 kg/m^3 is carried from ground level. Find the work done. The acceleration due to gravity is approximately 9.8m/s^2.
   Let y be the height of a slice measured from ground level. Then the radius of the slice using similar triangles is $r = 10 - \frac{4}{15}y$. Then the work done is
$$W = 17640\pi \int_0^{15} \left(10 - \frac{4}{15}y\right)^2 y dy = -17640\pi \int_{10}^6 r^2(10-r)\frac{15^2}{4^2} dr = \frac{496125\pi}{2}\int_6^{10} 10r^2 - r^3 dr = \frac{325458000\pi}{3}\text{Joules}$$