Math 125, Sections E and F, Midterm II

May 23, 2013

Name____

TA/Section______________________________________

Instructions.

- There are 4 questions. The exam is out of 40 points.
- You are allowed to use one page of notes written only on one side of the sheet in your own handwriting. **Hand in your notes with your exam paper.**
- You may use a calculator which does not graph and which is not programmable. Even if you have a calculator, give me exact answers. ($\frac{2\ln 3}{\pi}$ is exact, 0.7 is an approximation for the same number.)
- **Show your work**. If I cannot read or follow your work, I cannot grade it. You may not get full credit for a right answer if your answer is not justified by your work. If you continue at the back of a page, make a note for me. Please $\boxed{\text{BOX}}$ your final answer.

Question	points
1	
2	
3	
4	
Total	

1. Evaluate the following integrals.

(a) (5 points) $\int \frac{x^2+5x+7}{x+1}\,dx$

$u = x+1 \rightarrow x = u-1$

$du = dx$

$\int \frac{(u-1)^2+5(u-1)+7}{u}\,du$

$= \int \frac{u^2+3u+3}{u}\,du = \int u + 3 + \frac{3}{u}\,du$

$= \frac{u^2}{2} + 3u + 3\ln|u| + C$

$= \frac{(x+1)^2}{2} + 3(x+1) + 3\ln|x+1| + C$

OR

long division

$$\begin{array}{r|l} & x+4 \\ x+1 & x^2+5x+7 \\ & x^2+x \\ \hline & 4x+7 \\ & 4x+4 \\ \hline & 3 \end{array}$$

$\int x+4+\frac{3}{x+1}\,dx$

$= \frac{x^2}{2} + 4x + 3\ln|x+1| + C$

(b) (5 points) $\int_0^4 e^{\sqrt{x}}\,dx$

$w = \sqrt{x}$

$w^2 = x$

$2w\,dw = dx$

$\int_0^4 e^{\sqrt{x}}\,dx = \int_0^2 e^w 2w\,dw$

$u = 2w \qquad dv = e^w dw$

$du = 2dw \qquad v = e^w$

$= 2we^w\Big|_0^2 - \int_0^2 2e^w\,dw$

$= 4e^2 - 0 - 2e^w\Big|_0^2$

$= 4e^2 - 2e^2 + 2e^0$

$= 2e^2 + 2.$

2. Given the parametric curve

$$x = \frac{1}{2}t^2 \qquad y = \frac{4}{3}t^{3/2}, \qquad 0 \le t \le 4$$

(a) (2 points) Set up an integral to compute the length of the curve.

$$S = \int_0^4 \sqrt{\left(\frac{1}{2}\cdot 2t\right)^2 + \left(\frac{4}{3}\cdot\frac{3}{2}t^{1/2}\right)^2}\, dt = \int_0^4 \sqrt{t^2+4t}\, dt$$

(b) (5 points) Evaluate it to find the length of the curve.

complete square: $(t^2+4t+4)-4 = (t+2)^2-4$

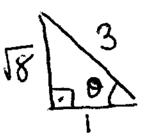
$t+2 = 2\sec\theta \qquad t=0 \to \sec\theta = 1 \to \theta = 0$

$dt = 2\sec\theta\tan\theta\, d\theta \qquad t=4 \to \sec\theta = 3 \to \theta = \sec^{-1}(3)$

$$\int_0^{\sec^{-1}(3)} (2\tan\theta)\, 2\sec\theta\tan\theta\, d\theta = \int_0^{\sec^{-1}3} 4\tan^2\theta\sec\theta\, d\theta = \int_0^{\sec^{-1}3} 4(\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= 4\int_0^{\sec^{-1}3} \sec^3\theta - \sec\theta\, d\theta = 4\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta\tan\theta| - \ln|\sec\theta + \tan\theta|\right]_0^{\sec^{-1}3}$$

$$= 2\cdot 3\cdot\sqrt{8} - 2\ln|3+\sqrt{8}|$$



$\sec\theta = 3$

$\tan\theta = \sqrt{8}$

(c) (3 points) Estimate the same integral in part (a) using Simpson's Rule with $n = 4$. Compare your answer with part (b) by computing the percentage error:

$$\text{percentage error} = \frac{\text{actual value} - \text{approximate value}}{\text{actual value}} \times 100\ \%$$

$f(t) = \sqrt{t^2+4t} \qquad \Delta t = \frac{4-0}{4} = 1$

$$S_4 = \frac{\Delta t}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right]$$

$$= \frac{1}{3}\left[\sqrt{0} + 4\sqrt{5} + 2\sqrt{12} + 4\sqrt{21} + \sqrt{32}\right]$$

$$\%\ \text{error} = \frac{6\sqrt{8} - 2\ln|3+\sqrt{8}| - \frac{1}{3}\left[4\sqrt{5} + 8\sqrt{3} + 4\sqrt{21} + 4\sqrt{2}\right]}{6\sqrt{8} - 2\ln|3+\sqrt{8}|} \times 100\ \%$$

$$\approx -\frac{2.151}{13.445} \approx 16\%$$

3. Determine if

$$\int_1^\infty \frac{x-1}{x(x+1)(x+2)}\,dx$$

converges by

(a) (3 points) Using the comparison test.

$$\frac{x-1}{x(x+1)(x+2)} \leqslant \frac{x}{x\cdot x\cdot x} = \frac{1}{x^2}$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges to $\frac{1}{2-1} = 1$, by comparison,

$$\int_1^\infty \frac{x-1}{x(x+1)(x+2)}\,dx \text{ converges.}$$

(b) (7 points) Evaluating the integral.

partial fractions

$$\frac{x-1}{x(x+1)(x+2)} = \frac{\frac{-1}{1\cdot 2}}{x} + \frac{\frac{-2}{-1\cdot 1}}{x+1} + \frac{\frac{-3}{-2\cdot(-1)}}{x+2}$$

$$\int_1^\infty \frac{x-1}{x(x+1)(x+2)}\,dx = \lim_{t\to\infty} \int_1^t \frac{-1/2}{x} + \frac{2}{x+1} - \frac{3/2}{x+2}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}\ln|x| + 2\ln|x+1| - \frac{3}{2}\ln|x+2| \Big|_1^t \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}\ln|t| + 2\ln|t+1| - \frac{3}{2}\ln|t+2| + \frac{1}{2}\ln|1| - 2\ln 2 + \frac{3}{2}\ln 3 \right]$$

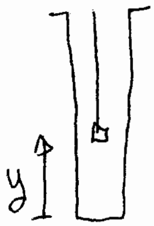
$$= \lim_{t\to\infty} \left[\frac{1}{2}\left(-\ln|t| + 4\ln|t+1| - 3\ln|t+2|\right) - 2\ln 2 + \frac{3}{2}\ln 3 \right]$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}\ln\left|\frac{(t+1)^4}{(t+2)^3 t}\right| + \frac{1}{2}\ln\left(\frac{27}{16}\right) \right]$$

$$= \frac{1}{2}\ln 1 + \frac{1}{2}\ln\left(\frac{27}{16}\right) = \frac{1}{2}\ln\left(\frac{27}{16}\right)$$

4. (10 points) You are pulling water out of a 30 meter deep well with a 2 kg leaky bucket tied to the end of a rope which has a mass of 0.4 kilograms per meter. Initially, the buckey scoops 10 kg of water but as you pull it up with a speed of 0.2 m/s, the bucket leaks at a rate of 0.05 kg/s.

   (a) When the bucket is y meters from the surface of the water in the well, how much water is left in the bucket? How much water reaces the top of the well?



it takes $\frac{y}{0.2}$ seconds to reach a height of y m.

So it looses $\left(\frac{y}{0.2}\right)(0.05) = 0.25y$ kg of water

So there is $10 - 0.25y$ kg left.

at the top of the well,

   (b) Find the work done in pulling the bucket to the top of the well. The acceleration due to gravity is 9.8 m/s.

$F(y)$ = weight of bucket + water + rope

$$= \Big(2 + (10 - 0.25y) + (30-y)(0.4) \Big) 9.8$$

$$W = \int_0^{30} \Big(2 + (10 - 0.25y) + (30-y)(0.4)\Big) 9.8 \, dy$$

$$= \int_0^{30} (24 - 0.65y) 9.8 \, dy = 9.8\left(24y - \frac{0.65y^2}{2}\right)\Bigg|_0^{30}$$

$$= 9.8\left(720 - \frac{585}{2}\right) = 4189.5 \text{ Joules.}$$