Math 125, Section I, Winter 2011, Solutions to Midterm II

1. Evaluate the following integrals.

   (a) First, a substitution

   $$w = \sqrt{x} \text{ or } w^2 = x \text{ so } 2wdw = dx$$

   then

   $$\int \sqrt{x}e^{\sqrt{x}}dx = 2\int w^2 e^w dw.$$

   Then you have to do integration by parts twice to get

   $$2\int w^2 e^w dw = 2w^2e^w - 4we^w + 4e^w + C = (2x - 4\sqrt{x} + 4)e^{\sqrt{x}} + C$$

   (b) First you complete the square in the denominator, then use substitution $u = x + 2$:

   $$\int \frac{x}{x^2 + 4x + 9}dx = \int \frac{x}{(x+2)^+5}dx = \int \frac{u-2}{u^2+5}du = \int \frac{u}{u^2+5}du - 2\int \frac{1}{u^2+5}du$$

   $$= \frac{1}{2}\ln|u^2+5| - \frac{2}{\sqrt{5}}\arctan\left(\frac{u}{\sqrt{5}}\right) + C = \frac{1}{2}\ln|x^2+4x+9| - \frac{2}{\sqrt{5}}\arctan\left(\frac{x+2}{\sqrt{5}}\right) + C$$

2. Evaluate the following integrals

   (a) First, complete the square. Then do the trig sub $x - 3 = \tan\theta$

   $$\int_3^{3+\sqrt{3}} \frac{x}{\sqrt{x^2-6x+10}} \quad dx = \int_3^{3+\sqrt{3}} \frac{x}{\sqrt{(x-3)^2+1}} \quad dx = \int_0^{\pi/3} 2\sec\theta + \sec\theta\tan\theta d\theta$$

   $$3\ln|\sec\theta + \tan\theta| + \sec\theta\Big|_0^{\pi/3} = 3\ln|2+\sqrt{3}| + 1$$

   (b) For this integral you need partial fractions:

   $$\int_2^4 \frac{x+1}{2x^3+x^2-3x}dx = \int_2^4 \frac{1/3}{x} + \frac{-2/15}{2x+3} + \frac{2/5}{x-1}dx$$

   $$= \frac{1}{3}\ln x - \frac{1}{15}\ln|2x+3| + \frac{2}{5}\ln|x-1|\Big|_2^4 = \frac{1}{3}\ln 2 - \frac{1}{15}\ln(11/7) + \frac{2}{5}\ln 3$$

3. The improper integral

$$\int_0^\infty e^{-x^3}\,dx$$

converges.

(a) Estimate the integral $\int_0^2 e^{-x^3}\,dx$ using Simpson's rule with $n = 6$. Give your answer with 3 digits after the decimal point.

$$\Delta x = \frac{1}{3}$$

$$S_6 = \frac{1}{9}\left[f(0) + 4f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + 4f\left(\frac{3}{3}\right) + 2f\left(\frac{4}{3}\right) + 4f\left(\frac{5}{3}\right) + f(2)\right] \approx 0.893$$

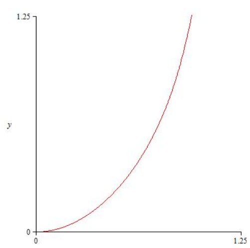
(b) (5 points) Show that the integral $\int_2^\infty e^{-x^3}\,dx$ converges by comparing it to an improper integral you can evaluate.

$$\int_2^\infty e^{-x^3}dx \le \int_2^\infty e^{-4x}dx = \lim_{t\to\infty}\int_2^t e^{-4x}dx = \lim_{t\to\infty}\left.\frac{e^{-4x}}{-4}\right|_2^t = \lim_{t\to\infty}\frac{e^{-4t}}{-4} + \frac{e^{-8}}{4} = \frac{1}{4e^8} \approx 0.00008$$

(c) (1 point) Estimate $\int_0^\infty e^{-x^3}\,dx$ using your results above. Give your answer with 3 digits after the decimal point.

$$\int_0^2 e^{-x^3}dx \approx 0.893$$

4. (10 points) The region bounded by the y-axis on the left and $y = \tan(x^2)$ on the right from $y = 0$ to $y = 1$ is rotated about the y axis to form a tank. The units on the axis are in meters. Find the work done to fill this container with seawater of density 1025 kg/m^3 pumped from the ground level at $y = 0$. Take the acceleration due to gravity to be 9.8 m/s^2. Below is the graph of $y = \tan(x^2)$ to get you started.



$$\text{Work} = \int_0^1 \pi\left(\sqrt{\arctan y}\right)^2 y(1025)(9.8)dy$$

The integral can be done using integration by parts with $u = \arctan y$ and $dv = ydy$:

$$\int_0^1 y\arctan y\,dy = \frac{y^2}{2}\arctan y\Big|_0^1 - \int_0^1 \frac{y^2/2}{y^2+1}dy = \frac{y^2}{2}\arctan y\Big|_0^1 - \frac{1}{2}\int_0^1 1 - \frac{1}{y^2+1}dy$$

$$= \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

so the work done is

$$(1025)(9.8)\left(\frac{\pi}{4} - \frac{1}{2}\right)\pi \approx 9006 \text{ Joules}$$