Math 125, Sections D and E, Midterm II, February 25, 2016

Name and Section Solutions

Instructions

- There are 4 questions. The exam is out of 40 points.
- You are allowed to use one page of notes written only on one side of the sheet in your own handwriting. Hand in you note sheet with your exam.
- You can use a Ti-30x IIS calculator. Put away all other electronic devices including smart watches.
- Leave all your answers in exact form. The expressions $e+1$, π^2 and $\sqrt{1+\ln 2}$ are exact. The decimals 3.718, 9.870 and 1.301 are approximations for the same numbers.
- For your integrals you may use the following formulas. Anything else must be justified by your work.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x| + C \qquad \int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \tan x\,dx = \ln|\sec x| + C$$

$$\int \csc x \cot x\,dx = -\csc x + C \qquad \int \csc x\,dx = \ln|\csc x - \cot x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C \qquad \int \frac{1}{a^2 - x^2}\,dx = \ln\left|\frac{x+a}{x-a}\right| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{dx}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

- **Show your work**. If we cannot read or follow your work, we cannot grade it. You may not get full credit for a right answer if your answer is not justified by your work. If you continue at the back of a page, make a note.

Question	points
1	
2	
3	
4	
Total	

1. Evaluate the following indefinite integrals.

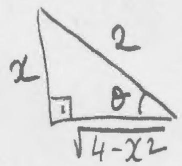
(a) (5 points) $\int \frac{x^2}{\sqrt{4-x^2}} dx$

$x = 2\sin\theta \longrightarrow \sqrt{4-x^2} = 2\cos\theta$

$dx = 2\cos\theta \, d\theta$

$$\int \frac{(2\sin\theta)^2}{2\cos\theta} \cdot 2\cos\theta \, d\theta = \int 4\sin^2\theta \, d\theta = \int 2(1-\cos 2\theta)\, d\theta$$

$$= 2\theta - \sin 2\theta + C = 2\theta - 2\sin\theta\cos\theta + C$$



so $\cos\theta = \frac{\sqrt{4-x^2}}{2}$, $\sin\theta = \frac{x}{2}$, $\theta = \arcsin\left(\frac{x}{2}\right)$

$$\int \frac{x^2}{\sqrt{4-x^2}} dx = 2\arcsin\left(\frac{x}{2}\right) - 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} + C$$

(b) (5 points) $\int 3xe^{2x} dx$

$u = 3x \qquad dv = e^{2x}$

$u = 3dx \qquad v = \frac{1}{2}e^{2x}$

$$\int 3xe^{2x} dx = \frac{3}{2}xe^{2x} - \int \frac{3}{2}e^{2x} dx$$

$$= \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + C$$

2. Evaluate the following integrals.

(a) (5 points) $\int_3^4 \frac{4-x}{x^3-3x^2+2x}dx = \int_3^4 \frac{4-x}{x(x-2)(x-1)}dx$

$$\frac{4-x}{x(x-2)(x-1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}$$

$$4-x = A(x-2)(x-1)+Bx(x-1)+Cx(x-2)$$

$x=0$ $\quad 4 = A(-2)(-1) \longrightarrow A=2$

$x=2$ $\quad 2 = B\cdot 2(2-1) \longrightarrow B=1$

$x=1$ $\quad 3 = C\cdot 1\cdot(-1) \longrightarrow C=-3$

$$\int_3^4 \frac{2}{x} + \frac{1}{x-2} - \frac{3}{x-1}\,dx = 2\ln|x| + \ln|x-2| - 3\ln|x-1|\Big|_3^4$$

$$= (2\ln 4 + \ln 2 - 3\ln 3) - (2\ln 3 + \ln 1 - 3\ln 2)$$

$$= 8\ln 2 - 5\ln 3$$

(b) (5 points) $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$

$$= \lim_{t\to\infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx \qquad u = \sqrt{x} \quad du = \frac{-1}{2\sqrt{x}}$$

$$= \lim_{t\to\infty} \int_{-1}^{-\sqrt{t}} -2e^u\,du$$

$$= -2\lim_{t\to\infty} e^u\Big|_{-1}^{-\sqrt{t}} = -2\lim_{t\to\infty}(e^{-\sqrt{t}} - e^{-1}) = -2(e^{-1}) = \frac{2}{e}.$$

3. (10 points) Let C be the curve

$$y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}, \qquad 1 \le x \le 4.$$

(a) (4 points) Set up an integral to compute the length of the curve.

$$y' = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$$

$$L = \int_1^4 \sqrt{1 + \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)}\, dx =$$

Do (b) **or** (c) below, **not both**. Circle the letter of the one you choose. If you attempt both, cross out the one you do not want us to grade.

(b) (6 points) Evaluate the integral to find the exact length of the curve.

(c) (6 points) Use Simpson's Rule with $n = 6$ to estimate the length of the curve.

$$\text{(b)}\ L = \int_1^4 \sqrt{1 + \frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}} = \int_1^4 \sqrt{\frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4}x^{-6}}\, dx$$

$$= \int_1^4 \sqrt{\left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2}\, dx = \int_1^4 \frac{1}{2}x^3 + \frac{1}{2}x^{-3}\, dx$$

$$\text{(b)}\ L = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^{-2}}{-2}\right]_1^4 = \frac{1}{2}\left[\left(64 - \frac{1}{32}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)\right] = \frac{1}{64}(2048 - 1 - 8 + 16)$$

$$= \frac{2055}{64} (\simeq 32.11)$$

$$\text{(c)}\ S_6 = \frac{0.5}{3}\left[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)\right]$$

where $f(x) = \sqrt{1 + \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2} = \frac{1}{2}(x^3 + x^{-3})$

$f(1) = \frac{1}{2}(1+1) = 1$

$4f(1.5) = 2(1.5^3 + 1.5^{-3}) \simeq 7.3426$

$2f(2) = 2^3 + 2^{-3} = 8.125$

$4f(2.5) = 2(2.5^3 + 2.5^{-3}) = 31.378$

$2f(3) = 3^3 + 3^{-3} \simeq 27.0370$

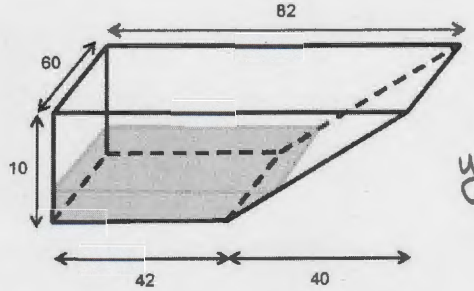
$4f(3.5) = 2(3.5^3 + 3.5^{-3}) \simeq 85.7966$

$f(4) = \frac{1}{2}(4^3 + 4^{-3}) \simeq 32.0078$

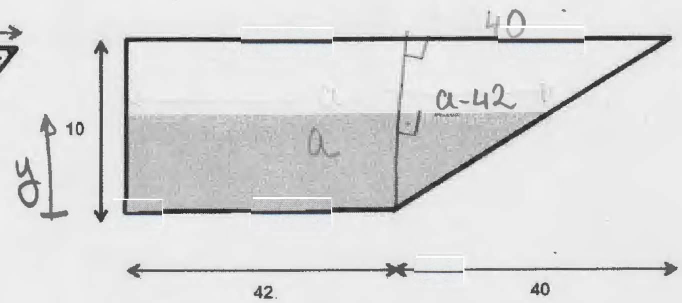
so $S_6 \simeq \frac{1}{6}(192.687)$

$\simeq 32.11$

4. A swimming pool is 60 feet wide and 82 feet long. It is 10 feet at its deep end. A cross section is shown below. Initially the pool is full. Find the work done in pumping all the water out of the pool through an outlet at the top of the pool. The weight density of water is 62.4 pounds per cubic feet.

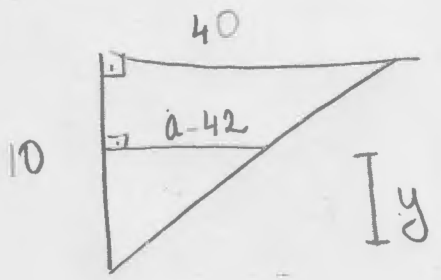


Cross section of the pool



Make sure you clearly mark your variable of integration on the picture so we can follow your solution.

$$\Delta \text{Work} = (60a)\Delta y\,(62.4)(10-y)$$



Similar Δs

$$\frac{40}{a-42} = \frac{10}{y}$$

$$40y = 10(a-42)$$

$$4y = a-42$$

$$4y+42 = a$$

$$W = \int_0^{10} 60(4y+42)(62.4)(10-y)\,dy$$

$$= 3744\int_0^{10} -4y^2-2y+420\,dy = 3744\left(-\frac{4y^3}{3}-y^2+420y\right)\Big|_0^{10}$$

$$= 3744\left(-\frac{4000}{3}-100+4200\right) = \frac{3744}{3}(-4000-300+12600)$$

$$= 10{,}358{,}400 \text{ ft. lbs}$$