## Math 126, Section C, Autumn 2012, Solutions to Midterm II

1. The vector function $\mathbf{r}(t) = \left\langle 4t+1, \frac{3}{2}t^2 - 7, 0\right\rangle$ traces a parabola on the $xy$-plane. Answer the following.

   (a) Compute $\mathbf{T}(t)$.

$$\mathbf{r}'(t) = \langle 4, 3t, 0\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{16+9t^2}$$

$$\mathbf{T}(t) = \left\langle \frac{4}{\sqrt{16+9t^2}}, \frac{3t}{\sqrt{16+9t^2}}, 0\right\rangle$$

   (b) Compute $\mathbf{N}(1)$.

$$\mathbf{T}'(t) = \left\langle -36t(16+9t^2)^{-3/2}, \frac{3(16+9t^2)^{1/2} - 27t^3(16+9t^2)^{-1/2}}{16+9t^2}, 0\right\rangle$$

$$\mathbf{T}'(1) = \left\langle -\frac{36}{125}, \frac{48}{125}, 0\right\rangle = \frac{12}{125}\langle -3, 4, 0\rangle$$

$$|\mathbf{T}'(1)| = \frac{12}{125}\sqrt{9+16+0} = \frac{12}{25}$$

$$\mathbf{N}(1) = \left\langle -\frac{3}{5}, \frac{4}{5}, 0\right\rangle$$

   (c) (1 point) What is $\mathbf{B}(t)$? Think before you compute.

$$\mathbf{B}(t) = \mathbf{k}$$

   (d) (2 points) Compute the curvature $\kappa(1)$

$$\kappa(1) = \frac{|\mathbf{T}'(1)|}{|\mathbf{r}'(1)|} = \frac{\frac{12}{25}}{\sqrt{25}} = \frac{12}{125}$$

   (e) (2 points) The osculating circle of this curve at the point $(5, -11/2, 0)$ (where $t = 1$) lies on the $xy$-plane. It has radius equal to the reciprocal of the curvature and its radius drawn from its center to the point of tangency is perpendicular to the curve. Find the equation of this oscullating circle. Write it in standard form $(x-a)^2 + (y-b)^2 = r^2$.

$$r = \frac{1}{\kappa(1)} = \frac{125}{12}$$

   If the vector from the point of tangency to the center is

$$\left\langle a-5, b+\frac{11}{2}, 0\right\rangle = r\mathbf{N}(1) = \left\langle -\frac{25}{4}, \frac{25}{3}, 0\right\rangle$$

   so $a = -5/4$ and $b = 17/6$ so the equation of the circle is

$$\left(x+\frac{5}{4}\right)^2 + \left(y-\frac{17}{6}\right)^2 = \frac{125^2}{144}$$

2. Given the equation $z^2y + 5zx - xy^3 = 5$

   (a) Use implicit differentiation to compute $z_x$ and $z_y$.

   Differentiate with respect to $x$

$$2zz_xy + 5z_xx + 5z - y^3 = 0$$

so

$$z_x = \frac{y^3 - 5z}{2zy + 5x}$$

Differentiate with respect to $y$

$$2zz_y + z^2 + 5z_y x - 3xy^2 = 0$$

so

$$z_y = \frac{3xy^2 - z^2}{2zy + 5x}$$

(b) Find the equation of the tangent line to the surface given by this equation at the point $(1, 1, 1)$.
At $(1, 1, 1)$ $z_x = -4/7$ and $z_y = 2/7$ so the tangent plane equation is

$$z - 1 = -\frac{4}{7}(x - 1) + \frac{2}{7}(y - 1)$$

(c) Use linear approximation to approximate the value of $c$ if $(0.95, 1.08, c)$ is on this surface.

$$c - 1 \approx -\frac{4}{7}(0.95 - 1) + \frac{2}{7}(1.08 - 1) = \frac{6.36}{7}$$

so

$$c \approx \frac{7.36}{7}$$

3. Find the points on the cone given by $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$. Make sure you verify that your points give minimum distance.
Instead of minimizing the distance, we can minimize the square of the distance:

$$d^2 = (x - 4)^2 + (y - 2)^2 + z^2$$

since $z^2 = x^2 + y^2$ the function to minimize is

$$f(x, y) = (x - 4)^2 + (y - 2)^2 + x^2 + y^2$$

Take the partial derivatives and set them equal to zero:

$$f_x = 2(x - 4) + 2x = 0$$

and

$$f_y = 2(y - 2) + 2y = 0$$

so $(x, y) = (2, 1)$ is the only critical point giving the points on the cone $(2, 1, \sqrt{5})$ and $(2, 1, -\sqrt{5})$. Last, we need to use the second derivatives to make sure this is a minimum:

$$f_{xx} = 4, f_{yy} = 4, fxy = 0$$

so $D = 16 > 0$ and $f_{xx} > 0$ so these points give minimum distances.

4. (10 points) Evaluate the integral

$$\int_0^1 \int_{5y}^5 e^{x^2} dx \quad dy$$

by reversing the order of integration.

$$\int_0^1 \int_{5y}^5 e^{x^2} dx \quad dy = \int_0^5 \int_0^{x/5} e^{x^2} dy \quad dx = \int_0^5 \frac{x}{5} e^{x^2} \quad dx = \frac{1}{10} e^{x^2} \Big|_0^5 = \frac{e^{25} - 1}{10}$$