Math 126, Section D, Winter 2010, Solutions to Midterm I

1. Given the two linear vector functions $\mathbf{r}_1(t) =< 2-t, 3+5t, 6t >$ and $\mathbf{r}_2(s) =< 3+s, 1+4s, -2+3s >$, answer the following questions about the lines they trace in space.

   (a) Show that the two lines are skew. That is they do not intersect and they are not parallel.

   They are not parallel because their direction vectors $v_1 =< -1, 5, 6 >$ and $v_2 =< 1, 4, 3 >$ are not: $< -1, 5, 6 >= a < 1, 4, 3 >$ has no solution because comparing first coordinates a must be -1 but then comparing second coordinates $5 = 4a = -4$ is wrong.

   They do not intersect because if we try to solve

   $$< 2-t, 3+5t, 6t >=< 3+s, 1+4s, -2+3s >$$

   we get $s = -1-t$ from the first component. Plugging that into the second we get $3+5t = 1+4(-1-t) = -3-4t$ so $t = -6/9$ and $s = -6/9-1 = -15/9$ than the last component gives $-36/9 = -63/9$ which is false.

   (b) Find the distance between them.

   First we need a vector normal to both vectors:

   $$\mathbf{n} =< -1, 5, 6 > \times < 1, 4, 3 >=< -9, 9, -9 >$$

   or it easier to work with the parallel vector $< -1, 1, -1 >$. Then the distance between the two lines is

   $$\left| \text{comp}_{<-1,1,-1>} \vec{P_1P_2} \right|$$

   where P_1 is a point on the first line and P_2 is a point on the second line. For example plug in 0 to both equations and $P_1 = (2, 3, 0)$ and $P_2 = (3, 1, -2)$ then $\vec{P_1P_2} =< 3-2, 1-3, -2-0 >=< 1, -2, -2 >$. Then,

   $$\text{comp}_{<-1,1,-1>} \vec{P_1P_2} = \frac{\vec{P_1P_2} \cdot < -1, 1, -1 >}{| < -1, 1, -1 > |} = \frac{1}{\sqrt{3}}.$$

   (c) The two skew lines lie on parallel planes. Find the equations of these two planes.(3 points)

   We already have a normal vector and the points from the previous part. So the first line is on the plane

   $$-(x-2) + (y-3) - z = 0$$

   and the first line is on the plane

   $$-(x-3) + (y-1) - (z+2) = 0$$

2. Write $< 2, 3, 5 >$ as a sum of two vectors $\mathbf{v}$ and $\mathbf{w}$; $\mathbf{v}$ parallel to $< 1, 2, -1 >$ and $\mathbf{w}$ normal to $< 1, 2, -1 >$. (6 points) The parallel one is

   $$\mathbf{v} = \mathbf{proj}_{<1,2,-1>} < 2, 3, 5 >= \frac{< 1, 2, -1 > \cdot < 2, 3, 5 >}{< 1, 2, -1 > \cdot < 1, 2, -1 >} < 1, 2, -1 >=< \frac{1}{2}, 1, -\frac{1}{2} >$$

   and then the normal one has to be

   $$\mathbf{w} =< 2, 3, 5 > -\mathbf{v} =< \frac{3}{2}, 2, \frac{11}{2} >$$

3. Given the vector function given by

$$\mathbf{r}(t) = < t^2 + 5, 3t^3 + 2 >$$

answer the following.

(a) Determine the concavity at the point $(6, -1)$. (6 points)
The concavity is determined by the value of $\frac{d^2y}{dx^2}$. The first derivative is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9t^2}{2t} = \frac{9}{2}t$$

then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\frac{9}{2}t}{2t} = \frac{\frac{9}{2}}{2t} = \frac{9}{4t}$$

The value of t that corresponds to the point $(6, -1)$ is $t = -1$ so the value of $\frac{d^2y}{dx^2}$ is -9/4 which is negative. The curve is concave down at that point.

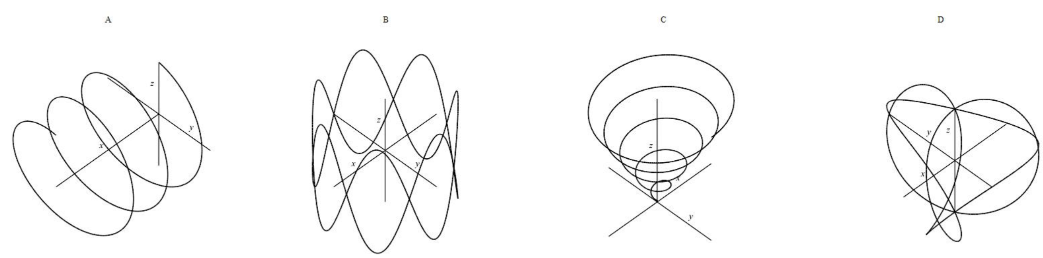
(b) Find the length of the curve for $0 \le t \le 2$.(2 points) The length is given by

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^2 \sqrt{(2t)^2 + (9t^2)^2} dt = \int_0^2 t\sqrt{4 + 81t^2} dt = \frac{328^{3/2} - 8}{243}$$

4. Answer the following.

(a) Match the following curves in space with their graphs by identifying the surface they are on. (8 points)

I. $x = t, y = \sin(3t), z = \cos(3t)$	Surface:$y^2 + z^2 = 1$	Graph:A
II. $x = t\sin(5t), y = t\cos(5t), z = t$	Surface:$x^2 + y^2 = z^2$	Graph: C
III. $x = \sin(2t), y = \cos(2t), z = \cos(7t)$	Surface:$x^2 + y^2 = 1$	Graph: B
IV. $x = \cos(t)\sin(3t), y = \sin(t)\sin(3t), z = \cos(3t)$	Surface:$x^2 + y^2 + z^2 = 1$	Graph:D



(b) Find parametric equations for the tangent line to $\mathbf{r}(t) = < \sin(2t), t^2 + 1, \ln(t + 1) >$ at the point $(0, 1, 0)$. (6 points)
A point on the line is $(0, 1, 0)$ which corresponds to $t = 0$ and a direction vector is given by $\mathbf{r}'(0)$.

$$\mathbf{r}'(t) = < 2\cos(2t), 2t, \frac{1}{t+1} >$$

so $\mathbf{r}'(0) = < 1, 0, 1 >$. Therefore the line is

$$\mathbf{r}_l(t) = < 0 + t, 1, 0 + t >$$

or

$$x = t, y = 1, z = t$$