Math 126, Section D, Winter 2010, Solutions to Midterm II

1. Answer the following questions about the vector function

$$\mathbf{r}(t) = \langle 3\sin(t), t, 3\cos(t)\rangle .$$

   (a) Find the length of the curve traced by this vector function from the point $(0, 0, 3)$ to the point $\left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}, \frac{3}{2}\right)$.

$$s = \int_0^{\pi/3} |\mathbf{r}'(t)|\, dt = \int_0^{\pi/3} \sqrt{(3\cos t)^2 + 1^2 + (-3\sin t)^2} dt == \int_0^{\pi/3} \sqrt{10} dt = \frac{\pi\sqrt{10}}{3}.$$

   (b) Find the curvature at the point $\left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}, \frac{3}{2}\right)$.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3} = \frac{|\langle 3\cos t, 1, -3\sin t\rangle \times \langle -3\sin t, 0, -3\cos t\rangle|}{|\langle 3\cos t, 1, -3\sin t\rangle|^3} = \frac{|\langle -3\cos t, 9, 3\sin t\rangle|}{10^{3/2}} = \frac{\sqrt{90}}{10^{3/2}} = \frac{3}{10}.$$

2. Evaluate the following integrals.

   (a)

$$\int\int x\left(x^2 + y^2\right)^{3/2} dA$$

   over the region R between the lines $y = x$, $y = \sqrt{3}x$ and the curve $y = \sqrt{9 - x^2}$.
   Here it is easier to use polar coordinates:

$$\int_{\pi/4}^{\pi/3} \int_0^3 (r\cos\theta) r^3 \cdot r dr d\theta = \left.\frac{r^6}{6}\right|_0^3 \cdot \left.\sin\theta\right|_{\pi/4}^{\pi/3} = \frac{243\left(\sqrt{3} - \sqrt{2}\right)}{4}$$

   (b)

$$\int_0^4 \int_{y/4}^1 y \ln\left(x^3 + 1\right) dx dy$$

   In this problem we have to change the order of integration. The region is inside the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 4)$.

$$\int_0^1 \int_0^{4x} y \ln\left(x^3 + 1\right) dy dx = \int_0^1 8x^2 \ln\left(x^3 + 1\right) dx = \frac{8}{3}\left.\left[\left(x^3 + 1\right)\ln\left(x^3 + 1\right) - \left(x^3 + 1\right)\right]\right|_0^1 = \frac{8(2\ln 2 - 1)}{3}$$

3. Let

$$f(x,y) = \left(\sqrt{x} + \sqrt{y}\right)^2$$

(a) Find the equation of the tangent plane to $z = f(x,y)$ at the point $(16, 100, 196)$.

$$f_x(x,y) = \frac{\left(\sqrt{x} + \sqrt{y}\right)}{\sqrt{x}} \qquad f_y(x,y) = \frac{\left(\sqrt{x} + \sqrt{y}\right)}{\sqrt{y}}$$

$$f_x(16,100) = \frac{\left(\sqrt{16} + \sqrt{100}\right)}{\sqrt{16}} = \frac{7}{2} \qquad f_y(16,100) = \frac{\left(\sqrt{16} + \sqrt{100}\right)}{\sqrt{100}} \frac{7}{5}$$

so the tangent plane has equation

$$z - 196 = \frac{7}{2}(x - 16) + \frac{7}{5}(y - 100)$$

(b) Approximate $\left(\sqrt{15} + \sqrt{99}\right)^2$ using linear approximation.

$$f(x,y) \approx 196 + \frac{7}{2}(x - 16) + \frac{7}{5}(y - 100)$$

so

$$f(15,99) \approx 196 + \frac{7}{2}(15 - 16) + \frac{7}{5}(99 - 100) = \frac{1911}{10} = 191.1$$

4. Find and classify all critical points of the function

$$f(x,y) = 3x - x^3 - 6xy^2.$$

Finding the critical points:

$$f_x(x,y) = 3 - 3x^2 - 6y^2 = 0 \qquad f_y(x,y) = -12xy = 0$$

From the second equation, $x = 0$ or $y = 0$. If $x = 0$, the first equation gives $y = \pm\frac{\sqrt{2}}{2}$ so $(0, \frac{\sqrt{2}}{2})$ and $(0, -\frac{\sqrt{2}}{2})$ are critical points. If $y = 0$, the first equation gives $3 - 3x^2 = 0$ so $x = \pm 1$ giving us the critical points $(1, 0)$ and $(-1, 0)$. To classify them we need the second order derivatives:

$$f_{xx}(x,y) = -6x \qquad f_{xy}(x,y) = -12y \qquad f_{yy}(x,y) = -12x$$

and the discriminant is

$$D(x,y) = 72x^2 - 144y^2.$$

At $(0, \frac{\sqrt{2}}{2})$, $D(0, \frac{\sqrt{2}}{2}) = -72$ so that point is a saddle.

At $(0, -\frac{\sqrt{2}}{2})$, $D(0, -\frac{\sqrt{2}}{2}) = -72$ so that point is also a saddle.

At $(1, 0)$, $D(1, 0) = 72$ and $f_{xx}(1, 0) = -6$ so it is a maximum.

At $(-1, 0)$, $D(-1, 0) = 72$ and $f_{xx}(-1, 0) = 6$ so it is a minimum.