Math 126, Sections A and B, Winter 2011, Solutions to Midterm I

1. Answer the following questions about the triangle with vertices $A(1,4,5)$, $B(1,8,8)$ and $C(3,6,5)$.

   (a) Find the angle A.

$$\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{\left|\vec{AB}\right|\left|\vec{AC}\right|} = \frac{<0,4,3>\cdot<2,2,0>}{\sqrt{16+9}\sqrt{4+4}} = \frac{8}{5\sqrt{8}} = \frac{\sqrt{8}}{5}$$

   so $\theta = \cos^{-1}\left(\frac{\sqrt{8}}{5}\right)$.

   (b) Draw a line from the point A perpendicular to the side BC. Call the point where this line intersects BC point D. Find the coordinats of point D. (3 points)

$$\vec{BD} = \mathbf{proj}_{\vec{BC}}\vec{BA} = \frac{\vec{BA}\cdot\vec{BC}}{\vec{BC}\cdot\vec{BC}}\vec{BC} = \frac{<0,-4,-3>\cdot<2,-2,-3>}{<2,-2,-3>\cdot<2,-2,-3>}<2,-2,-3>=<2,-2,-3>$$

   so $D = C = (3,6,5)$. The angle C is 90 degrees.

   (c) Find the area of the triangle.

   You can use

$$\text{Area} = \frac{1}{2}|\vec{BC}||\vec{AD}|$$

   or use the cross product, for example,

$$\text{Area} = \frac{1}{2}|\vec{AB}\times\vec{AC}|$$

   In any case, you should get $\sqrt{136}/2$.

2. The line l_1 is perpendicular the plane $2x+3y+z=24$ at the point $(4,5,1)$. The line l_2 is the line passing through the points $(0,2,0)$ and $(6,11,3)$.

   (a) Find the vector equation for the line l_1.

   The direction vector for the line is the same as the normal vector for the plane so

$$\mathbf{r}_1(t) =< 4+2t, 5+3t, 1+t > .$$

   (b) Find the vector equation for the line l_2.

$$\mathbf{r}_2(s) =< 6s, 2+9s, 3s > .$$

   (c) Are the two lines the same, skew, parallel or intersecting?

   The have parallel direction vectors so they may be parallel. Since they are not the same (for example th epoint $(4,5,1)$ on the first line is not on the second one), they must be parallel.

3. Let C be the curved traced by the vector function $\mathbf{r}(t) = \langle 2\cos t - \sin t, \sin t, \cos t\rangle$.

(a) Find two surfaces so C is their intersection. Use your surfaces to sketch and describe the shape of the curve.

The curve lies on the surfaces $y^2 + z^2 = 1$ which is a circular right cylinder with the x axis running through its center and $2z - y - x$ which is a plane. So the curve looks like an ellipse.

(b) Set up and integral to find the length of the curve you have above. Do not integrate. Since

$$\mathbf{r}'(t) =< -2\sin t - \cos t, \cos t, -\sin t >$$

the arclength is given by

$$s = \int_0^{2\pi} \sqrt{(-2\sin t - \cos t)^2 + (\cos t)^2 + (-\sin t)^2} dt$$

(c) Find the equation of the tangent line to the curve at the point where $t = \pi/4$.

$$\mathbf{r}'\left(\frac{\pi}{4}\right) =< -2\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right), \cos\left(\frac{\pi}{4}\right), -\sin\left(\frac{\pi}{4}\right) >= \frac{\sqrt{2}}{2} < -3, 1, -1 >$$

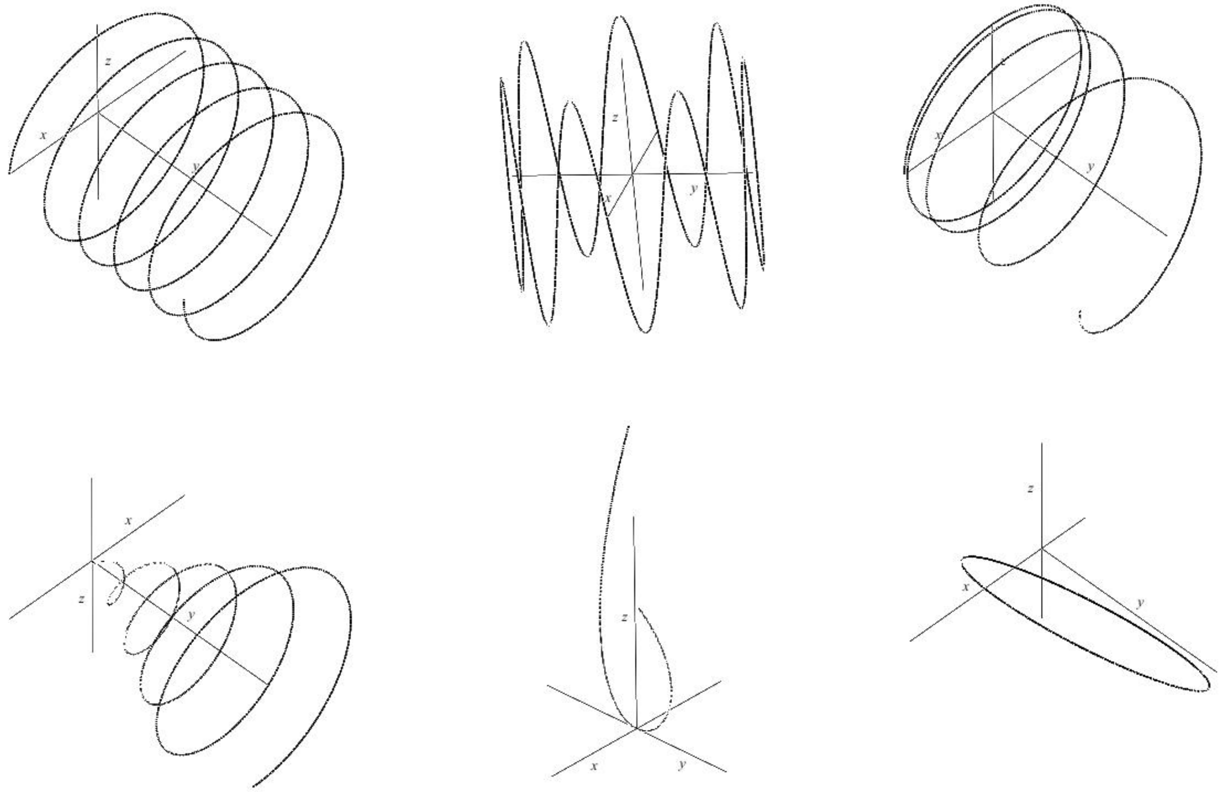
so the tangent line is given by

$$\mathbf{r}_1(t) = \left\langle \frac{\sqrt{2}}{2} - 3t, \frac{\sqrt{2}}{2} + t, \frac{\sqrt{2}}{2} - t \right\rangle.$$

4. (a) Match the following vector equations by the curves below. In all graphs, the z axis points up. (6 points)

$$\mathbf{r_1}(t) = \langle \sin t, \cos t, \cos 7t \rangle \qquad \mathbf{r_2}(t) = \langle 4t\cos t, t, 4t\sin t \rangle \qquad \mathbf{r_3}(t) = \langle 2\cos t + 1, \sin t + 2, 5\cos t + 1 \rangle$$

$$\mathbf{r_4}(t) = \langle 4\cos t, t, 4\sin t \rangle \qquad \mathbf{r_5}(t) = \left\langle t^3, 5t, 2t^2 \right\rangle \qquad \mathbf{r_6}(t) = \left\langle 4\cos t, t^3, 4\sin t \right\rangle$$



The pictures correspond to curves top row:4, 1, 6, and bottom row: 2, 5, 3.

(b) Decide if the following are True or False. You do not need to explain your answer. (4 points)

1. False If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions then $\frac{d}{dt}\left(\mathbf{u}(t) \times \mathbf{v}(t)\right) = \mathbf{u}'(t) \times \mathbf{v}'(t)$.

2. True If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

3. False If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

4. True For any two vectors $\mathbf{u}$ and $\mathbf{v}$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.