Math 126, Sections C and D, Winter 2014, Midterm I

January 28, 2014

Name Key

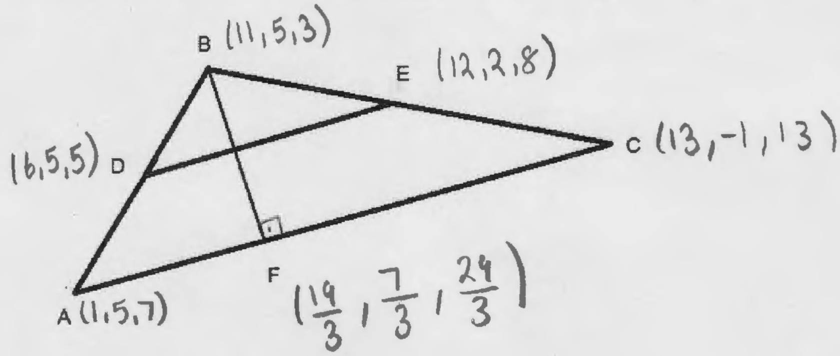
TA/Section

Instructions.

- There are 4 questions. The exam is out of 40 points.
- You are allowed to use one page of notes written only on one side of the sheet in your own handwriting. **Hand in your notes with your exam paper.**
- You may use a calculator which does not graph and which is not programmable. Even if you have a calculator, give me exact answers. ($\frac{2\ln 3}{\pi}$ is exact, 0.7 is an approximation for the same number.)
- **Show your work**. If I cannot read or follow your work, I cannot grade it. You may not get full credit for a right answer if your answer is not justified by your work. If you continue at the back of a page, make a note for me. Please $\boxed{\text{BOX}}$ your final answer.

Question	points
1	
2	
3	
4	
Total	

1. In the triangle below, we know the coordinates of the points $A(1,5,7)$, $B(11,5,3)$ and $E(12,2,8)$. The point D is the midpoint of the line segment AB and the point E is the midpoint of the line segment BC. The line BF is perpendicular to the side AC.



(a) (2 points) Find the coordinates of the point D.

$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\langle 11-1, 5-5, 3-7\rangle = \frac{1}{2}\langle 10,0,-4\rangle = \langle 5,0,-2\rangle$

Since $A=(1,5,7)$, D is $(5+1, 0+5, -2+7) = (6,5,5)$

(b) (2 points) Find the coordinates of the point C.

$\overrightarrow{BC} = 2\overrightarrow{BE} = 2\langle 12-11, 2-5, 8-3\rangle = 2\langle 1,-3,5\rangle = \langle 2,-6,10\rangle$

Since $B=(11,5,3)$, C is $(2+11, -6+5, 10+3) = (13,-1,13)$

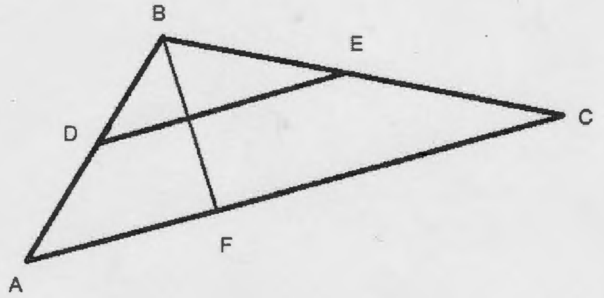
(c) (3 points) Find the coordinates of the point F.

$$\overrightarrow{AF} = \text{proj}_{\overrightarrow{AC}}\overrightarrow{AB} = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\overrightarrow{AC}\cdot\overrightarrow{AC}}\overrightarrow{AC} = \frac{\langle 10,0,-4\rangle\cdot\langle 12,-6,6\rangle}{\langle 12,-6,6\rangle\cdot\langle 12,-6,6\rangle}\langle 12,-6,6\rangle$$

$$= \frac{120+0-24}{144+36+36}\langle 12,-6,6\rangle = \frac{96}{216}\cdot 6\langle 2,-1,1\rangle = \frac{8}{3}\langle 2,-1,1\rangle = \left\langle \frac{16}{3}, -\frac{8}{3}, \frac{8}{3}\right\rangle$$

So $F = \left(\frac{16}{3}+1, -\frac{8}{3}+5, \frac{8}{3}+7\right) = \left(\frac{19}{3}, \frac{7}{3}, \frac{29}{3}\right)$

$A(1,5,7)$, $B(11,5,3)$, $E(12,2,8)$



(d) (3 points) What is the area of the triangle?

$$\text{Area} = \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}|\vec{BA}\times\vec{BC}| = \frac{1}{2}|\vec{CA}\times\vec{CB}| = \frac{1}{2}|\vec{AC}||\vec{BF}|$$

computing the first one:

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 10 & 0 & -4 \\ 12 & -6 & 6 \end{vmatrix} = (0-24)\vec{i} - (60+48)\vec{j} + (-60)\vec{k}$$

$$= \langle -24, -108, -60\rangle = -12\langle 2, 9, 5\rangle$$

$$|\vec{AB}\times\vec{AC}| = 12\sqrt{4+81+25} = 12\sqrt{110}$$

$$\text{Area} = \frac{12}{2}\sqrt{110} = 6\sqrt{110}$$

(e) (2 points) Is the line DE parallel to the line AC?

$$\vec{DE} = \langle 12-6, 2-5, 8-5\rangle = \langle 6, -3, 3\rangle$$

$$\vec{AC} = \langle 12, -6, 6\rangle = 2\langle 6, -3, 3\rangle \quad \text{Yes.}$$

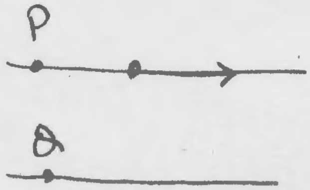
2. (10 points) Find the equation of the plane which contains the two parallel lines

$$x = 3 + 2t \qquad y = 7 - 2t \qquad z = 6 + 4t$$

and

$$x = -4 + t \qquad y = 12 - t \qquad z = 2 + 2t.$$

When you are done, verify that your plane contains the two lines.



$\vec{v} = \langle 1, -1, 2 \rangle$

P $t=0$ $(3, 7, 6)$ $\qquad \vec{PQ} = \langle -7, 5, -4 \rangle$

Q $t=0$ $(-4, 12, 2)$

$$\vec{n} = \vec{PQ} \times \vec{v} = \begin{vmatrix} i & j & k \\ -7 & 5 & -4 \\ 1 & -1 & 2 \end{vmatrix} = (10-4)\vec{i} - (-14+4)\vec{j} + (7-5)\vec{k}$$

$$= \langle 6, 10, 2 \rangle = 2\langle 3, 5, 1 \rangle$$

Plane equation:

$$3(x-3) + 5(y-7) + 1\cdot(z-6) = 0$$

$$3x + 5y + z = 9 + 35 + 6 = 50$$

$$\boxed{3x + 5y + z = 50}$$

Verifying:

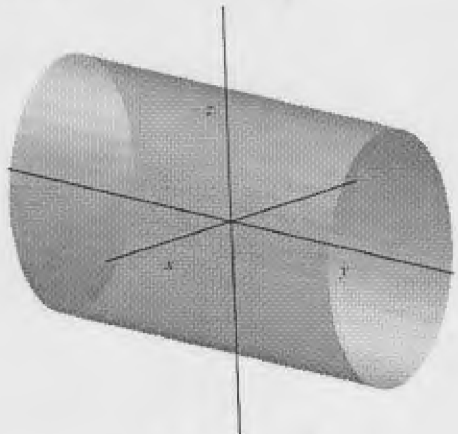
$$3(3+2t) + 5(7-2t) + (6+4t) = 9 + 6t + 35 - 10t + 6 + 4t = 50$$

$$3(-4+t) + 5(12-t) + (2+2t) = -12 + 3t + 60 - 5t + 2 + 2t = 50$$

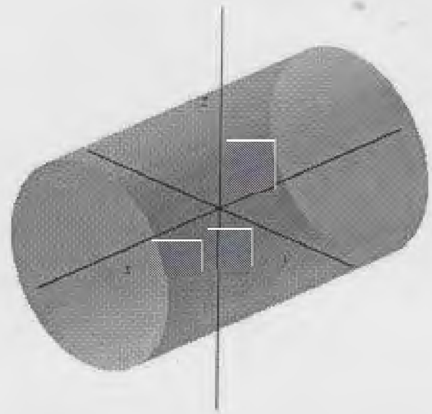
3. (a) (4 points) The curve traced by the vector function

$$\mathbf{r(t)} = \langle \mathbf{2\sin(t), t, 3\cos(t)} \rangle$$

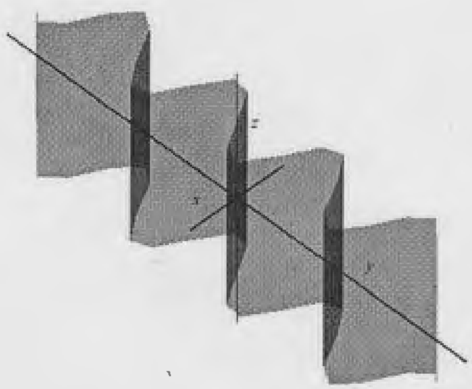
is contained in three of the four surfaces sketched below. Write down the equations of the three surfaces which DO contain the curve and put an X under the surface which does NOT contain the curve.



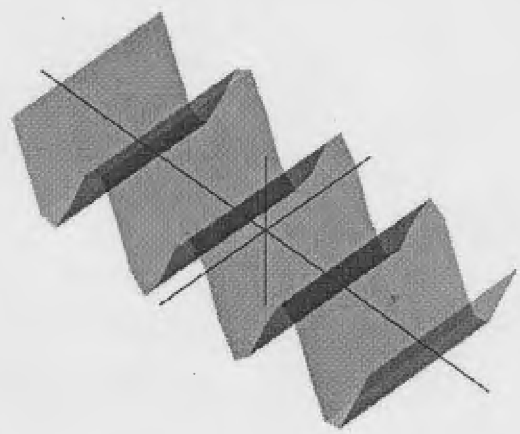
$\left(\frac{x}{2}\right)^2+\left(\frac{z}{3}\right)^2=1$



X



$x=2\sin y$



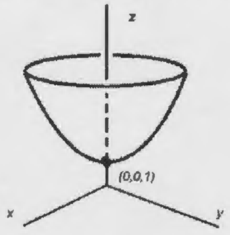
$z=3\cos y$

(b) (4 points) Identify the surface given by the equation

$$4x^2 - 24x + 100y^2 - 25z^2 - 50z = 89.$$

Use the terminology of surfaces from Section 12.6. Either sketch the surface or descrobe its orientation.

For example, write "$x^2 + y^2 = 5z - 5$ is an elliptic paraboloid which opens up in the positive z direction with its lowest point at (0,0,1)" or sketch



elliptic paraboloid

completing squares

$$4(x^2-6x+9)+100y^2-25(z^2+2z+1)=89+36-25$$

$$4(x-3)^2+100y^2-25(z+1)^2=100$$

$$\frac{(x-3)^2}{5^2}+y^2-\frac{(z+1)^2}{2^2}=1$$

Hyperboloid of one sheet with "center" at (3,0,-1)

It opens as you go up or down the z direction

OR

hyperboloid of one sheet

4. Given the curve

$$x = \cos(2\pi t) \qquad y = \sin(2\pi t) \qquad z = 3 - t^{3/2},$$

(a) (5 points) Find the parametric equations of the tangent line to the curve at the point when $t = 4$.

$\vec{r}(t) = \langle \cos(2\pi t), \sin(2\pi t), 3 - t^{3/2} \rangle$

$\vec{r}(4) = \langle \cos 8\pi, \sin 8\pi, 3 - 4^{3/2} \rangle = \langle 1, 0, -5 \rangle$

$\vec{r}\,'(t) = \langle -2\pi \sin(2\pi t), 2\pi \cos(2\pi t), -\frac{3}{2} t^{1/2} \rangle$

$\vec{r}\,'(4) = \langle -2\pi \sin 8\pi, 2\pi \cos 8\pi, -\frac{3}{2}\sqrt{4} \rangle = \langle 0, 2\pi, -3 \rangle$

Tangent Line:

$\vec{r}(t) = \langle 1, 0, -5 \rangle + t\langle 0, 2\pi, -3 \rangle = \langle 1, 2\pi t, -5-3t \rangle$

$x = 1 \qquad y = 2\pi t \qquad z = -5-3t$

(b) (5 points) Compute the length of the curve from the point where $t = 0$ to the point where $t = 8$.

$$\int_0^8 \sqrt{(-2\pi \sin 2\pi t)^2 + (2\pi \cos 2\pi t)^2 + \left(-\frac{3}{2}\sqrt{t}\right)^2}\, dt$$

$$= \int_0^8 \sqrt{4\pi^2 + \frac{9}{4}t}\, dt \qquad u = 4\pi^2 + \frac{9}{4}t \qquad du = \frac{9}{4}\, dt$$

$$= \frac{4}{9}\int_{4\pi^2}^{4\pi^2+18} \sqrt{u}\, du = \frac{4}{9}\,\frac{2}{3}\, u^{3/2}\Big|_{4\pi^2}^{4\pi^2+18} = \frac{8}{27}\left((4\pi^2+18)^{3/2} - 8\pi^3\right)$$