Solutions to Math 126 Winter 2020 MT1

1. (8 points) The line through the points $P(2, 4, -2)$ and $Q(2, -1, 3)$ is tangent to the sphere with center $C(1, 0, -2)$ and radius 3. Find the equation of the sphere, vector equation for the line, and the point of tangency.

Sphere: $(x-1)^2 + y^2 + (z+2)^2 = 9$

line vector $\vec{PQ} = \langle 0, -5, 5 \rangle$ (or $\langle 0, -1, 1 \rangle$)

point $P(2, 4, -2)$ (or $Q(2, -1, 3)$)

$\vec{r}(t) = \langle 2, 4-t, -2+t \rangle$

Finding point of tangency

intersecting the line with the sphere

$(2-1)^2 + (4-t)^2 + (-2+t+2)^2 = 9$

$1 + 16 - 8t + t^2 + t^2 = 9$

$2t^2 - 8t + 8 = 0$

$t^2 - 4t + 4 = 0$

$(t-2)^2 = 0$

$t = 2$

OR

using the fact that at the point of tangency R $\vec{CR} \perp \langle 0, -1, 1 \rangle$

$R(2, 4-t, -2+t)$ since its on the line

$\vec{CR} = \langle 2-1, 4-t, -2+t+2 \rangle = \langle 1, 4-t, t \rangle$

$\vec{CR} \cdot \langle 0, -1, 1 \rangle = 0 - 4 + t + t = 0$

$2t = 4$

$t = 2$

In any case $\vec{r}(2) = \langle 2, 2, 0 \rangle$
so the point of tangency is $(2, 2, 0)$

2. (10 points) The two parts of this question are not related.

(a) Determine if the following are True or False. You do not have to justify your answers.

- **T** **F** Two lines parallel to the same plane are parallel.
- **T** **F** For any two vectors $\mathbf{v}$ and $\mathbf{w}$, we always have $\mathbf{v} \cdot \mathbf{w} \times \mathbf{v} = 0$.
- **T** **F** The point $P(1, -3, 4)$ is on the plane $2x - y + z = 9$.

  $2(1) - (-3) + 4 = 2 + 3 + 4 = 9$
- **T** **F** The lines $\mathbf{r_1}(t) = \langle 2-t, 1+3t, 5+2t \rangle$ and $\mathbf{r_2}(t) = \langle 4+2t, -5-t, 1+3t \rangle$ are parallel.

  $\vec{v_1} = \langle -1, 3, 2 \rangle$ $\quad \vec{v_2} = \langle 2, -1, 3 \rangle$
- **T** **F** The points $P(1, 0, 3)$, $Q(3, 2, 1)$ and $R(0, 3, -2)$ are on the same line.

  $\vec{PQ} = \langle 2, 2, -2 \rangle$ $\quad \vec{PR} = \langle -1, 3, -5 \rangle$
- **T** **F** The line $\mathbf{r}(t) = \langle 1+3t, -3t, 1-2t \rangle$ is on the plane $4x + 2y + 3z = 7$.

  $4(1+3t) + 2(-3t) + 3(1-2t) = 4 + 12t - 6t + 3 - 6t = 7$

(b) Under each picture, write down the name of the curve, the equation of the surface graphed with the curve and the name of the surface. For example, $\mathbf{r_5}$, $x = 7$, plane. You have to get all three right to get the point for that part. The z-axis points up in all pictures.

$\mathbf{r_1}(t) = \langle 2\cos(t), 3\sin(t), 5\sin(8t) \rangle$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\mathbf{r_2}(t) = \langle 2\cos(8t), 3\sin(8t), 5\sin(t) \rangle$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\mathbf{r_3}(t) = \langle 2t\cos(t), 3t^2 + 2, 2t\sin(t) \rangle$

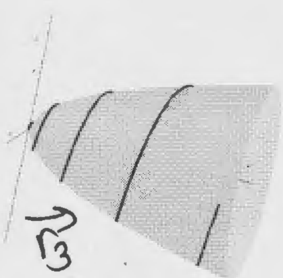
$x^2 + z^2 = 4t^2$

$\left(\frac{x}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = \frac{y-2}{3}$

$\mathbf{r_4}(t) = \langle 2\sin(2t)\cos(9t), \sin(9t), 3\cos(2t)\cos(9t) \rangle$

$\left(\frac{x}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = \cos^2(9t)$

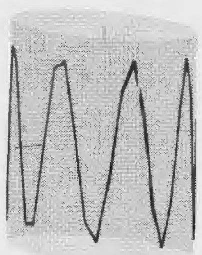
$\left(\frac{x}{2}\right)^2 + y^2 + \left(\frac{z}{3}\right)^2 = 1$



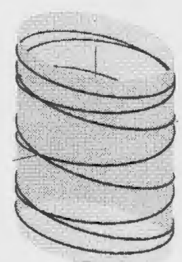
$\vec{r_3}$

$\frac{x^2}{4} + \frac{z^2}{4} = \frac{y-2}{3}$

(elliptic) paraboloid



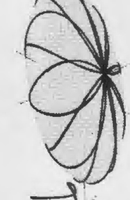
$\vec{r_1}$



$\vec{r_2}$

BOTH $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

(elliptic) cylinder



$\vec{r_4}$

$\left(\frac{x}{2}\right)^2 + y^2 + \left(\frac{z}{3}\right)^2 = 1$

ellipsoid

3. (10 points) The following questions are about the triangle with vertices $A(1,0,3)$, $B(2,5,-1)$ and $C(-3,2,6)$.

(a) Find the angle at vertex A.

$\overrightarrow{AB} = \langle 1, 5, -4 \rangle$

$\overrightarrow{AC} = \langle -4, 2, 3 \rangle$

$$\cos\theta = \frac{-4+10-12}{\sqrt{1+25+16}\,\sqrt{16+4+9}} = \frac{-6}{\sqrt{42}\,\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{-6}{\sqrt{42}\sqrt{29}}\right) = \pi - \cos^{-1}\left(\frac{6}{\sqrt{42}\,\sqrt{29}}\right)$$

(b) Find the equation of the plane containing the triangle. Give your answer in standard form.

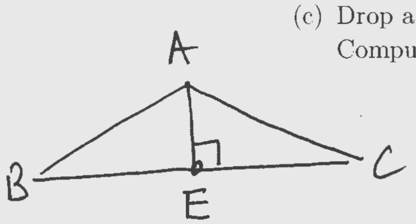
$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -4 \\ -4 & 2 & 3 \end{vmatrix} = (15+8)\vec{i} - (3-16)\vec{j} + (2+20)\vec{k}$$

$$= \langle 23, 13, 22 \rangle$$

$$23(x-1) + 13(y-0) + 22(z-3) = 0$$

$$23x + 13y + 22z = 89$$

(c) Drop a perpendicular from vertex A to side BC. Call the point where it intersects BC point E. Compute the vector $\vec{BE}$.



$$\overrightarrow{BE} = \overrightarrow{\text{proj}}_{\overrightarrow{BC}}\, \overrightarrow{BA}$$

$\overrightarrow{BA} = \langle -1, -5, 4 \rangle$

$\overrightarrow{BC} = \langle -5, -3, 7 \rangle$

$$= \frac{5+15+28}{25+9+49} \langle -5, -3, 7 \rangle = \frac{48}{83} \langle -5, -3, 7 \rangle$$

$$= \left\langle \frac{-240}{83}, \frac{-144}{83}, \frac{336}{83} \right\rangle$$

4. (12 points) Given the vector function $\mathbf{r}(t) = \langle t^2 \sin t, 2t, t^2 \cos t \rangle$ answer the following questions about the curve it traces in space.

(a) Compute the length of the curve from $t = -\pi$ to $t = \pi$.

$$\vec{r}\,'(t) = \langle 2t\sin t + t^2\cos t,\ 2,\ 2t\cos t - t^2 \sin t \rangle$$

$$|\vec{r}\,'| = \sqrt{4t^2\sin^2 t + 4t^3\cancel{\sin t}\cos t + t^4\cos^2 t + 4 + 4t^2\cos^2 t - 4t^3\cancel{\sin t\cos t} + t^4\sin^2 t}$$

$$= \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$$

$$\text{arclength} = \int_{-\pi}^{\pi} t^2 + 2\, dt = \frac{t^3}{3} + 2t \Big|_{-\pi}^{\pi} = 2\left(\frac{\pi^3}{3} + 2\pi\right)$$

(b) Find the vector equation of the tangent line to the curve at the point where $t = \pi$.

$$\vec{r}(\pi) = \langle 0, 2\pi, -\pi^2 \rangle \qquad \vec{r}\,'(\pi) = \langle -\pi^2, 2, -2\pi \rangle$$

$$\vec{r}_t(t) = \langle -\pi^2 t, 2\pi + 2t, -\pi^2 - 2\pi t \rangle$$

(c) Find the curvature of the point at the point where $t = 0$.

$$\vec{r}\,'' = \langle 2\sin t + 2t\cos t + 2t\cos t - t^2\sin t,\ 0,\ 2\cos t - 2t\sin t - 2t\sin t - t^2\cos t \rangle$$

$$\vec{r}\,''(0) = \langle 0, 0, 2 \rangle \qquad \vec{r}\,'(0) = \langle 0, 2, 0 \rangle$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \langle 4, 0, 0 \rangle$$

$$K = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{4}{2^3} = \frac{1}{2}$$