Math 126 Solutions to Midterm II

1. (a) $f_x(Q) \approx \dfrac{\Delta z}{\Delta x} \approx 3.5,\ f_y(P) \approx \dfrac{\Delta z}{\Delta y} \approx 1.2,\ f_{xx}(Q) < 0,\ f_{yy}(P) > 0$

   (b) Local Minimum : $(-1.3, 0.5)$, $(1.1, 0.5)$
   Local Maximum : $(0.2, -1.2)$
   Saddle : $(0.2, 0.5)$, $(1.1, -1.2)$, $(-1.3, -1.2)$

2. (a) Differentiate with respect to x:

$$4xy + 6yzz_x + 4y^2 = 0$$

   at $(1,1,1)$, $4 + 6z_x + 4 = 0$, so $z_x = -4/3$.
   Differentiate with respect to x again:

$$4y + 6yz_xz_x + 6yzz_{xx} = 0$$

   at $(1,1,1)$, $z_x = -4/3$ so $z_{xx} = -22/9$.

   Differentiate with respect to y:

$$2x^2 + 3z^2 + 6yzz_y + 8xy = 0$$

   at $(1,1,1)$, $2 + 3 + 6z_y + 8 = 0$ so $z_y = -13/6$.

   (b) $f(x,y) = \ln(1+xy) + y^2 + 3x$.

$$f_x(x,y) = \frac{y}{1+xy} + 3 \qquad f_y(x,y) = \frac{x}{1+xy} + 2y$$

$$f_x(0,1) = \frac{1}{1} + 3 = 4 \qquad f_y(0,1) = \frac{0}{1} + 2 = 2$$

   Tangent Plane

$$z = 1 + 4(x-0) + 2(y-1)$$

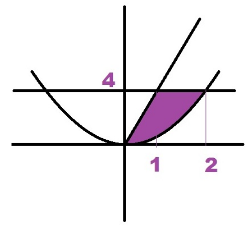
   Linear Approximation

$$f(x,y) \approx 1 + 4(x-0) + 2(y-1)$$

   so

$$f(0.05, 0.97) \approx 1 + 4(0.05-0) + 2(0.97-1) = 1.14$$

3. (a) .



(b)

$$\int_0^4 \int_{y/4}^{\sqrt{y}} xy + y^2 \, dx \, dy$$

(c)

$$\int_0^1 \int_{x^2}^{4x} xy + y^2 \, dy \, dx + \int_1^2 \int_{x^2}^{4} xy + y^2 \, dy \, dx$$

(d)

$$\int_0^4 \int_{y/4}^{\sqrt{y}} xy + y^2 \, dx \, dy = \int_0^4 \frac{x^2 y}{2} + xy^2 \Big|_{y/4}^{\sqrt{y}} dy = \int_0^4 \frac{y^2}{2} + y^{5/2} - \frac{9}{32} y^3 \, dy = \frac{614}{21}.$$

4. **Volume of Ellipsoid:**

$$\iint_{x^2+y^2\leq 1} 2\sqrt{\frac{1-x^2-y^2}{4}} = \int_0^{2\pi} \int_0^1 \sqrt{1-r^2}\, r \, dr \, d\theta = 2\pi \frac{2}{3} (1-r^2)^{3/2} \left(-\frac{1}{2}\right) \Big|_0^1 = \frac{2\pi}{3}$$

Volume of Box:
Let (x, y, z) be the corner of the box in the first octant. Then $V = 8xyz$. Usind the constraint $x^2 + y^2 + 4z^2 = 1$ (solving for z) we get

$$f(x, y) = 4xy\sqrt{1-x^2-y^2}.$$

Then,

$$f_x = 4y\sqrt{1-x^2-y^2} - \frac{4x^2 y}{\sqrt{1-x^2-y^2}} = 4y \left[\sqrt{1-x^2-y^2} - \frac{x}{\sqrt{1-x^2-y^2}} \right] = 0$$

gives $x = \sqrt{1-x^2-y^2}$ or $1 = 2x^2 + y^2$. By symmetry, $f_y = 0$ gives $1 = x^2 + 2y^2$ so

$$x = y = \frac{1}{\sqrt{3}} \qquad \textit{and} \qquad z = \frac{\sqrt{1-x^2-y^2}}{2} = \frac{1}{2\sqrt{3}}$$

so the volume of the box is is $\dfrac{4}{3\sqrt{3}}$.

RATIO: $\dfrac{\sqrt{3}}{2}\pi$.