Math 120 Final Review

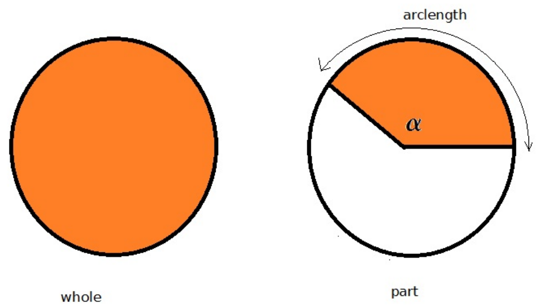
Part A - Mechanics

This section summarizes the math tools we talked about in Chapters 15-20 after the second midterm. You may put some of the material in this part on your note sheet.

Circles and Angles

A circle of radius r has

$$\text{AREA} = \pi r^2 \qquad\qquad \text{CIRCUMFERENCE} = 2\pi r.$$



One revolution of the circle is 360° or 2π radians. So in order to compute the area of a slice or the length of an arc we use ratios:

For α° (in degrees)

$$\text{Area} = \frac{\alpha^\circ}{360^\circ}\pi r^2 \qquad\qquad \text{Arclength} = \frac{\alpha^\circ}{360^\circ}2\pi r$$

For α in radians

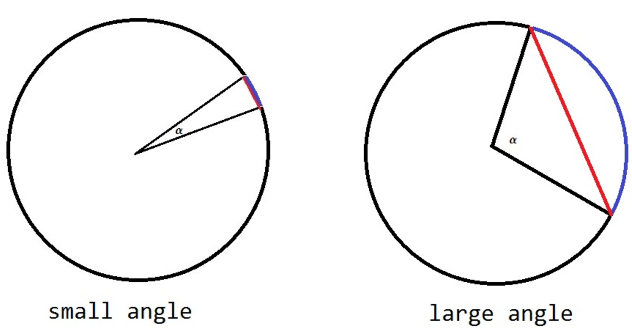
$$\text{Area} = \frac{\alpha}{2\pi}\pi r^2 \qquad\qquad \text{Arclength} = \frac{\alpha}{2\pi}2\pi r.$$

The arclength formula on the right for the angle α in **radians** simplifies to

$$\mathbf{Arclength} = \boldsymbol{\alpha} \cdot \mathbf{r}$$

which is simple and **very** useful. In particular, this is what we use to relate angular speed to linear speed below.

If the angle is **small**, you can approximate the length of a chord (red in pictures) by the length of the arc (blue in pictures). The chord length would use the coordinates of the two endpoints and the lengthy distance formula with a square root whereas the arclength formula above is very simple to use.



$$\text{Chord length} \approx \alpha \cdot r$$

when α is small and in radians. If the angle α is in degrees, the chord length is still approximated by the arclength , but you need to change the arclength formula on the right.

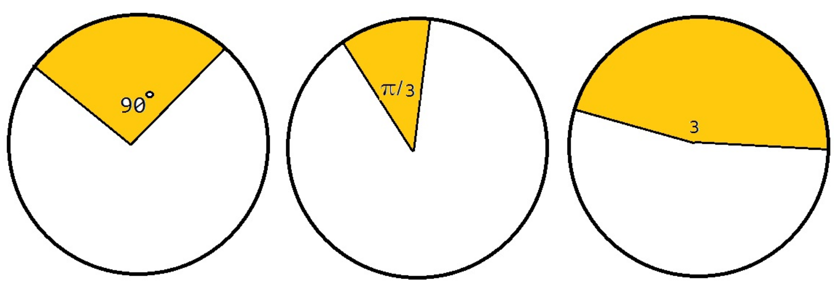
Problems in this first section are for you to practice the mechanics so you can comfortably use them in longer problems. Make sure you can do all the exercises in this part before you attempt long problems. The final exam questions will not be like these. They will be more likely be like the questions from Part B below. However, each will use skills and formulas from this part. Successfull problem solving is like making a puzzle: You break up the problem into small steps/pieces until each piece is familiar and simple to solve. Then, you put them together for the complete solution. A succesfull problem solver is one who is able to break up the big problem until she recognizes the small and familiar pieces.

The default angle measure is **radians**. If the little degree circle is missing, the angle is in radians.

1. Do the conversions to complete the table:

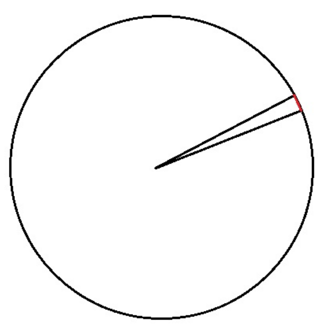
Revolutions	1				$\frac{1}{6}$			0.37
Radians	2π	π		$\frac{\pi}{6}$		5		
Degrees	$360°$		$90°$				$72°$	

2. Find the area of the slice and the length of the arc in the pictures below.



A , radius=7 B ,radius=4 C ,radius=12.35

3. Estimate the length of the chord (in red) if the radius is 3.4 cm and the slice is $\frac{1}{24}^{\text{th}}$ of the circle.



Motion Around a Circle

When something moves around a circular path, there are two ways to measure its speed:

(a) **Linear Speed**: This is the usual distance covered in a unit of time. We will use the letter v (for velocity) since the letter s is traditionally used for arclength.

(b) **Angular Speed**: This describes the angle covered per unit of time. For the angle, we can use radians, degrees or revolutions (rotations).

To convert between angular speed ω and linear speed v, which have the **same time unit** and the angle in **radians**, we use

$$v = \omega \cdot r$$

which follows from

$$\text{Arclength} = \alpha \cdot r$$

since v is arclength covered per unit time and ω is angle covered per unit time.
A good way to practice theses conversions are belt and wheel problems. A belt around two wheels transfers linear speed, i.e. two wheels connected by a belt have the same linear speed, that of the belt. If two wheels are fixed at their centers through a common axle, they will turn together, i. e. they will have the same angular speed.

4. Practice your unit conversions:

   32 ft/second = inches/sec = ft/min = inches/hr

   There are 12 inches in a foot.

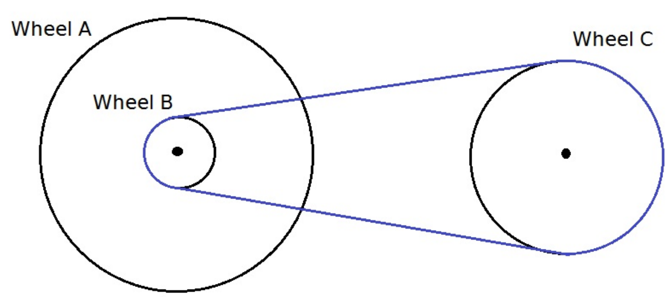
5. Practice your unit conversions:

   2 rad/min = °/min = rev/sec = rev/hr

6. If a unicycle wheel of radius 52 centimeters rotates at 2.5 revolutions per second, what is the linear speed of the unicycle on the road?

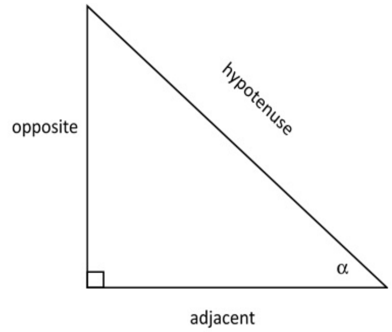
7. If Copper runs around a circular lake of radius 2.5 km. at 18 km/hr, what is his angular speed in radians per minute?

8. Wheels A and B are fixed together at the axle. Wheels B and C are connected by a belt. Wheel A has radius 8 cm, Wheel B 3 cm, and Wheel C 6 cm. Find the angular and linear speeds of all 3 wheels in the picture below if Wheel C is rotating at 5 revolutions per minute. Use radians/minute for the angular speeds and centimeters per minute for the linear speeds.



Sine, Cosine and Tangent

For an acute angle θ, i.e. $0 < \theta < \frac{\pi}{2}$ or $0° < \theta° < 90°$, we define $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ as ratios on a right triangle. By similar triangles, these ratios do not depend on the particular right triangle chosen with the given angle α.



$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}}$$

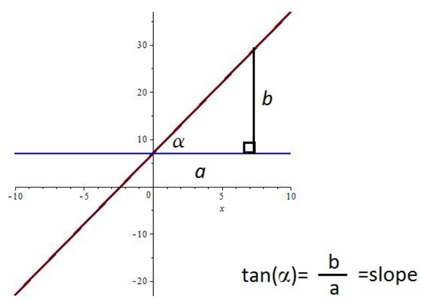
$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}}$$

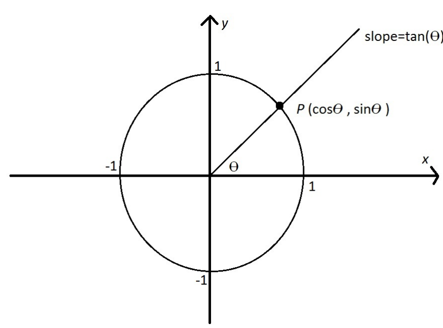
In particular, if the right triangle has hypotenuse 1, then the legs are $\sin(\alpha)$ and $\cos(\alpha)$. Then, from the Pythagorean Theorem we get the very important trigonometric identity

$$\cos^2(\alpha) + \sin^2(\alpha) = 1$$

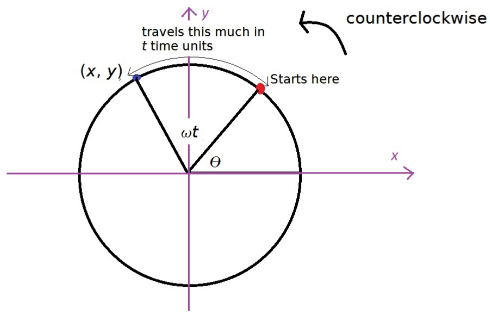
The slope of a line is the tangent value of its angle of inclination.



We then extend the definition of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ to all real numbers, except for some domian restrictions on the tangent function. Start measuring the angle in radians from the positive x-axis in the counterclockwise direction. Mark the point P where the other side of the angle intersects the unit circle. The point P has coordinates $(\cos(\theta), \sin(\theta))$ and the line through the origin and point P has slope $\tan(\theta)$. Note that the tangent values are not defined when the line is vertical, at values $\frac{\pi}{2} + 2n\pi$ for any integer n. That is when the point P falls on the y-axis.



If there is an object going around a circular path of radius r, we can write parametric equations for its position at a given time t. Assuming the center of the circular path is on the origin and the object moves counterclockwise.



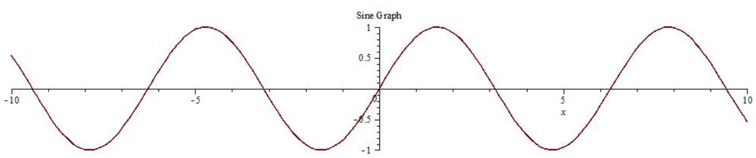
Then the equations are

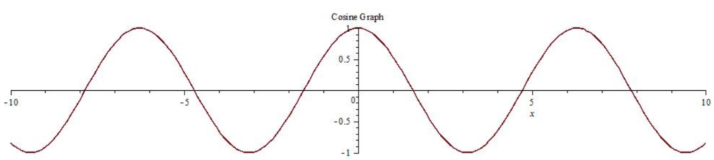
$$x = r\cos(\omega t + \theta) \qquad\qquad y = r\sin(\omega t + \theta)$$

9. Copper starts running counterclockwise at the northmost point of a circular lake of radius 5 kilometers at 17 meters per socond. Find his coordinates 6 minutes after he starts running by imposing a coordinate system where the center of the circle is at the origin.

Trigonometric Functions and Sinosuidal Functions

From the definitions of sine, cosine and tangent on the unit circle, we get the following graphs of the functions $z = \sin(\theta)$, $z = \cos(\theta)$ and $z = \tan(\theta)$.

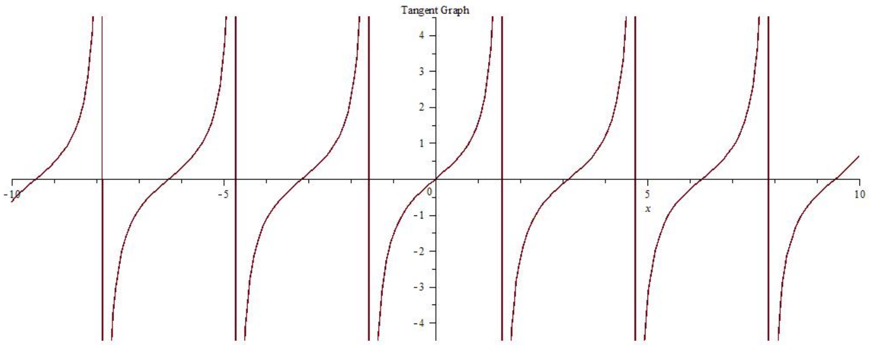




The sine and cosine graphs have exactly the same shape. They both have period 2π. One is a shifted version of the other.

$$\cos(\theta) = \sin\left(\theta - \frac{\pi}{2}\right)$$

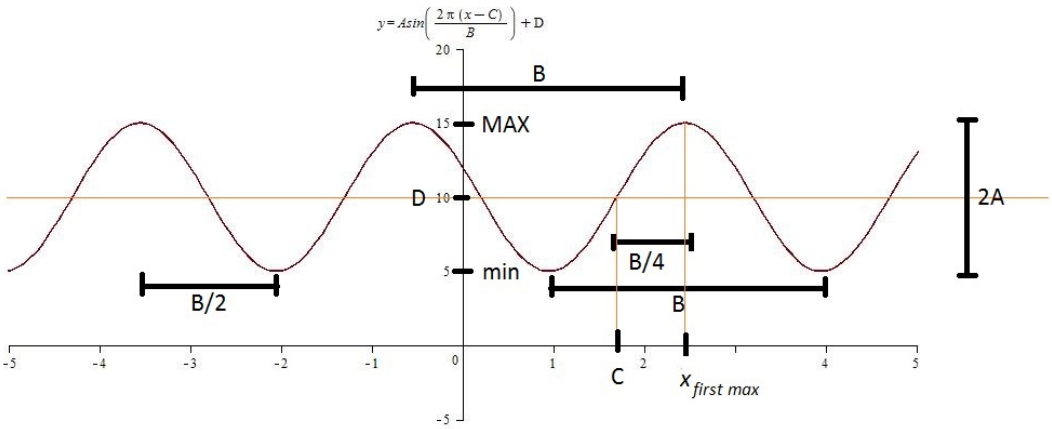
The tangent graph has vertical asymptotes at the θ values where is is not defined. The period of the tangent function is π.



In modelling, we prefer to use the sine function. After vertical and horizontal streches and vertical and horizontal shifts, a sinosuidal function has the standard form

$$y = A \sin\left(\frac{2\pi}{B}(t - C)\right) + D$$

and the graph looks like



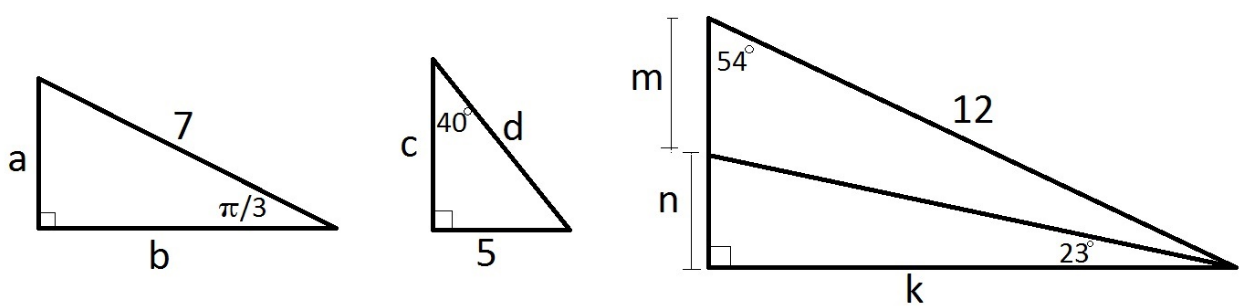
From a story, we usually figure out the amplitude A and the average value or the y-shift D from

$$A = \frac{\text{MAX} - \text{min}}{2} \qquad\qquad D = \frac{\text{MAX} + \text{min}}{2}.$$

The value B is the period. It is the time it takes for a complete cycle. You can compute it by looking at the horizontal distance between two max values or between two min values. It is also twice the horizontal distance between the min and next the max value. The x-shift C can be computed by

$$C = x_{\text{MAX}} - \frac{B}{4}$$

10. Find the missing sides of the triangles in the pictures below.



11. Put $y = 3\cos(5x - 7) + 2$ in standard form and find the constants A, B, C, D.

12. A sinusoidal function $f(x)$ is increasing at $x = 0$ until it reaches its maximum value of $f(1) = 12$. It then decreases towards f(6)=7. Find the constants A, B, C, D amd write down the function in standard from.

Inverse Trigonometric Functions

None of the three trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$ is one to one. To define their inverses, we first restrict their domains. Then, define their inverses:

$$y = \sin^{-1}(x) \qquad \text{means} \qquad x = \sin(y) \text{ and } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$y = \cos^{-1}(x) \qquad \text{means} \qquad x = \cos(y) \text{ and } 0 \le x \le \pi$$

$$y = \tan^{-1}(x) \qquad \text{means} \qquad x = \tan(y) \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

We also use $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$ for the inverses.

In order to solve an equation of the form $\sin(x) = a$, we first find one solution $x = \sin^{-1}(a)$. Then, since $\sin(x) = \sin(\pi - x)$ from the unit circle by symmetry, we find the other $x = \pi - \sin^{-1}(a)$. Then since the sine function has period 2π, all the solutions are given by

$$x = \sin^{-1}(a) + 2n\pi \qquad \text{or} \qquad x = \pi - \sin^{-1}(a) + 2n\pi$$

where n is any integer. In a problem, there may be restrictions on the values of x so only finitely many of these are acceptable.

13. What is $\sin^{-1}(1)$. Do not use a calculator.
14. Find all x values with $\sin(x) = 0.3$.
15. Find all $-1 < x < 7$ with $\sin(2x) = 0.25$.

Part B- Problems

The final questions will be like the problems in this section. You can find the old finals and their solutions at http://www.math.washington.edu/∼m120/.The questions below are grouped into topics although one problem may be using several ideas. The questions below are all from final exams.

Circles and Circular motion

1. Winter 2015, Question 4
2. Winter 2015, Question 5
3. Autumn 2014, Question 5
4. Autumn 2014, Question 7

Trigonometric functions and their inverses

1. Winter 2015, Question 1
2. Autumn 2014, Question 1
3. Autumn 2014, Question 8
4. Spring 2014, Question 6
5. Winter 2014, Question 8

Once you go through these, look at other old final exams for more questions. Do at least one complete exam to test your timing skills. Remember that the final exam is cumulative so you have to review Chapters 1-14 as well.

Answers to Part A Problems

1. .

Revolutions	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{5}{2\pi}$	$\frac{1}{5}$	0.37
Radians	2π	π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	5	$\frac{2\pi}{5}$	0.74π
Degrees	360°	180°	90°	30°	60°	$\frac{900}{\pi}^\circ$	72°	133.2°

2. A. Area=$\frac{49\pi}{4}$, arclength=3.5π. B. Area=$\frac{8\pi}{3}$, arclength=$\frac{4\pi}{3}$. C. Area=228.78375, arclength=37.05.

3. Chord≈arclength=$\frac{17}{60}\pi$.

4. 32 ft/second = 384 inches/sec = 1920 ft/min = 1382400 inches/hr

5. 2 rad/min = $\frac{360}{\pi}^\circ$/min = $\frac{1}{60\pi}$ rev/sec = $\frac{60}{\pi}$ rev/hr

6. 260π centimeters per second.

7. 0.12 radians per minute.

8. $\omega_C = 10\pi$ rad/min, $v_C = 60\pi$ cm/min $=v_B$, $\omega_B = 20\pi$ rad/min$=\omega_A$, $v_A = 160\pi$ cm/min.

9. $x = 5000\cos(1.224 + 0.5\pi) \approx -4702.33$ meters. $y = 5000\sin(1.226 + 0.5\pi) \approx 1699.43$. He is at a point Northwest of the center of the lake.

10. $a = \frac{7\sqrt{3}}{2}$, $b = \frac{7}{2}$, $d = \frac{5}{\sin(40^\circ)} \approx 7.779$, $c = \frac{5}{\tan(40^\circ)} \approx 5.959$, $k = 12\sin(54^\circ) \approx 9.708$, $n = 12\sin(54^\circ)\tan(23^\circ) \approx 4.121$, $m = 12\cos(54^\circ) - 12\sin(54^\circ)\tan(23^\circ) \approx 2.933$.

11. A sinusoidal function $f(x)$ starts at $f(0) = 8.3$ and increases until it reaches its maximum value of $f(3) = 12$. Its minimum value is 7. Find the constants A, B, C, D amd write down the function in standard from.

12. $\sin^{-1}(1) = \frac{\pi}{2}$.

13. $\sin^{-1}(0.3) + 2n\pi$, $\pi - \sin^{-1}(0.3) + 2n\pi$ where n is any integer.

14. The five approximate possible values of x are: 0.126, 1.445, 3.268, 4.586, 6.410.