Math 120 Midterm 1 Review

Part A - Mechanics

This section summarizes the math tools we talked about in the first seven chapters. You may want to put some or all of the formulas on your note sheet. It may help to accompany the formulas with relative pictures.

Geometry: Similar triangles, the Pythagorean Theorem and the distance formula

For practice on similar triangles see the Triangles worksheet at
http://www.math.washington.edu/ ebekyel/Math120/Triangles.pdf.

The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The distance formula follows from the Pythagorean Theorem when you make a right triangle with legs of length $|y_2 - y_1|$ and $|x_2 - x_1|$ and the hypotenuse connecting the two points. Draw a picture to see this.

Problems in this first section are for you to practice the mechanics so you can comfortably use them in longer story problems. If you are sure about how you would solve a question, skip it. If you are not sure, solve it and see if you get it right. The midterm questions will not be like these. They will be more likely be like the questions from Part B below, with stories. But remember, you need to practice your scales before you attempt to play Chopin on the piano.

1. What is the distance between the points $(1, 2)$ and $(-3, 11)$.

   Lines and circles

   The equation of a line with slope m and point (x_1, y_1) is

   $$y - y_1 = m(x - x_1).$$

   We usually simplify it and write in in the form $y = mx + b$, where b is the y- intercept. If you know two points on the line (x_1, y_1) and (x_2, y_2) you can compute the slope by

   $$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

   The product of the slopes of two perpendicular lines is -1. In order to find the shortest distance from a point P to a line l you have to draw a perpendicular line segment from the point P to the line l and calculate its length.

   The standard equation of the circle is

   $$(x - h)^2 + (y - k)^2 = r^2$$

   where (h, k) is the center of the circle and r is its radius. By completing the squares- both for x and y- in an equation of the type

   $$x^2 + Ax + y^2 + Cy + D = 0$$

   you can see the standard form of the circle equation and read its center and radius.

   To intersect a line $y = mx + b$ with a circle $(x - h)^2 + (y - k)^2 = r^2$ you solve for x and y by first eliminating one of the variables. There may be 0,1 or 2 solutions.

2. Find the equation of the line with slope 5 passing through the point $(-1, 2)$.

3. Find the equation of the line passing through the points $(2, 10)$ and $(\frac{1}{2}, 0)$. What is the y-intercept? What is the slope?

4. The line l_1 through the point $(-1, 2)$ intersects the line l_2 through $(3, 7)$ and $(6, -1)$ with a right angle. What are the equations of the lines and what is the point of their intersection?

5. The area of the triangle with sides given by $y = mx$, $y = -3x + 11$ and the x-axis is 12. What is the value of m?

6. Find the distance from $(9, 1)$ to the line $y = 2x - 11$. (See Question 4)

7. Find the center and the radius of the circle given by the equation
$$x^2 + 12x + y^2 + 7y = 1$$
and sketch its graph.

8. Problems about lines and circles:
   (a) Find the points of intersection of the horizontal line $y = 1$ with the circle $x^2 + y^2 = 5^2$. What is the distance between the the points of intersection?
   (b) Find the points of intersection of the vertical line $x = 3$ with the circle $(x - 2)^2 + (y + 1)^2 = 5$. What is the distance between the two points of intersection?
   (c) Find the equations of the two lines passing through the point $(3, 0)$ which are tangent to the circle $x^2 + y^2 = 1$.

Parametric equations

If a particle travels on a **line** on the coordinate plane with **constant speed** then its coordinates (x, y) as a function of time t are given by
$$x = a + bt \qquad\qquad y = c + dt$$
where a, b, c, d are numbers. To find the coefficients a, b, c, d, you need to know where the particle is at two different times, usually at $t = 0$ and some other time.

9. If a particle starts at the point $(2, 5)$ and travels in a straight path at constant speed to reach the point $(11, -7)$ at $t = 3$, what are its equations of motion? What is its speed?

10. If a particle travels from $(2, -1)$ to $(10, 2)$ on a straight line at a constant speed of 3 units per second, how long does it take the particle to reach the point $(10, 20)$? What are its equations of motion as a function af t where t is the time elapsed in seconds since it has left the point $(2, -1)$.

Multi-part functions

A multi-part function is one which has different equations for parts of its domain. The most important multi-part function is the absolute value function.

11. Sketch a graph of the multi-part function
$$f(x) = \begin{cases} \frac{1}{5}x + 4 & \text{if} \quad x < 5 \\ -\sqrt{1 - (x - 6)^2} + 5 & \text{if} \quad 5 \le x \le 7 \\ -2x + 19 & \text{if} \quad x > 7 \end{cases}$$
and use the graph to solve for x in the equation $f(x) = 3$. Hint: The graph of the equation with the square root is part of a circle.

12. Solve for x in
$$x + 2 = 3 - |2x + 1|.$$

13. Algebra practice:

    (a) Solve for y in terms of x in
$$\frac{3}{5-4y} = x.$$

    (b) Solve for α in terms of β in
$$\frac{3}{\alpha+3} = 2\beta - 5.$$

    (c) Solve for x in
$$\frac{3}{2+\frac{1}{2-\frac{1}{x}}} = 12$$

Parabolas

Any equation of the form
$$y = ax^2 + bx + c$$
with $a \neq 0$ has the graph of a parabola. To see what the parabola looks like, complete the square on the right and bring the equation to the form
$$y = a(x-h)^2 + k.$$
The point (h,k) is the vertex of the parabola. If $a > 0$, the parabola opens up and at $x = \frac{-b}{2a}$ the function $f(x) = ax^2 + bx + c$ has its minimum value $f(-b/2a)$. If $a < 0$ then the parabola opens down and at $x = h = \frac{-b}{2a}$ the function $f(x) = ax^2 + bx + c$ has its maximum value $f(-b/2a)$. So $h = -\frac{b}{2a}$ and $k = f(-b/2a)$.

The values of x where $ax^2 + bx + c = 0$ are called the roots or zeros. They are the x intercepts of the graph of the function $y = f(x)$. Sketch three parabolas with two x-intercepts, one x-intercept, no x intercepts.
To solve the equation
$$ax^2 + bx + c = 0$$
we use the **quadratic formula**
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
It is a consequence of completing the square in $ax^2 + bx + c = 0$, but we used it so frequently, it is better just to memorize it instead of completing the square each time.

14. Find the points of intersection line through $(11, 21)$ and $(16, 31)$ with the circle $x^2 + y^2 = 16$. What is the distance between the two points of intersection?

15. Compete the square in
$$y = -5x^2 + 15x + 7$$
and sketch the parabola. Look at your sketch to see how many roots it has and use the quadratic formula to find them, if any.

16. Find the point(s) of intersection of the line $y = 2x - 6$ with the parabola $y = -x^2 + 1$, if any. First, sketch them both to verify how many points of intersection there are.

17. Solve for x in
$$\sqrt{2x+11} = 7 - \sqrt{x-3}$$
by first eliminating the square roots, then solving the resulting equation and finally checking your answers with the original equation.

Part B- Problems

The midterm questions will be like the problems in this section. You can find the old midterms and their solutions at http://www.math.washington.edu/∼m120/.They are grouped into topics although one problem may be using several ideas. Once you go through these, look at other old midterms for more questions. Do at least one complete exam to test your timing skills.

Circles and lines

1. Winter 15, Ostroff, Question 1
2. Spring 14, Ostroff, Question 3
3. Autumn 14, Conroy, Question 1

Parametric equations

1. Winter 15, Ostroff, Question 2
2. Winter 14, Pezzoli, Question 1
3. Autumn 13, Ostroff, Question 3

Multi-part functions

1. Winter 13, Conroy, Question 3
2. Autumn 14, Ostroff, Question 3
3. Spring 14, Ostroff, Question 2
4. Autumn 13, Ostroff, Question 4

Quadratic Functions

1. Winter 15, Ostroff, Question 4
2. Spring 13, Conroy, Question 4
3. Winter 14, Pezzoli, Question 1
4. Autumn 13, Ostroff, Question 4

Algebra

1. Autumn 14, Conroy, Question 2
2. Autumn 13, Conroy Question 4

Answers to Part A Problems

1. $\sqrt{(-3-1)^2+(11-2)^2}=\sqrt{97}$.

2. $y-2=5(x+1)$ or $y=5x+7$.

3. The slope $m=\frac{10-0}{2-\frac{1}{2}}=\frac{20}{3}$ so the equation is $y=\frac{20}{3}\left(x-\frac{1}{2}\right)$ or $y=\frac{20}{3}x-\frac{10}{3}$.

4. $m_2=\frac{7-(-1)}{3-6}=-\frac{8}{3}$ and $m_1=\frac{3}{8}$ so l_1 has equation $y=\frac{3}{8}x+\frac{19}{8}$ and l_2 has equation $y=-\frac{8}{3}x+15$.They intersect at $(x,y)=\left(\frac{303}{71},\frac{1129}{248}\right)$.

5. The two lines intersect at $(x,y)=\left(\frac{11}{m+3},\frac{11m}{m+3}\right)$ so the triangle has area $\frac{111m}{6(m+3)}$ which gives $m\frac{216}{49}$.

6. The line through $(9,1)$ which is perpendicular to $y=2x-11$ has equation $y=-\frac{1}{2}x+\frac{11}{2}$ and the two intersect at $(x,y)=\left(\frac{33}{5},\frac{11}{5}\right)$ so the distance is $\sqrt{\left(\frac{33}{5}-9\right)^2+\left(\frac{11}{5}-1\right)^2}=\frac{6}{\sqrt{5}}$.

7. $(x+6)^2+\left(y+\frac{7}{2}\right)^2=\left(\frac{\sqrt{197}}{2}\right)^2$.

8. (a) The points are $(x,y)=\left(\pm\sqrt{24},1\right)$ and the distance is $2\sqrt{24}$.
   (b) The points are $(x,y)=(3,1)$ and $(x,y)=(3,-3)$and the distance is 4.
   (c) The points of tangency are $(a,b)=\left(\frac{1}{3},\pm\frac{\sqrt{8}}{3}\right)$, the slopes are $\pm\frac{1}{8}$ so the line equations are $y=-\frac{1}{\sqrt{8}}(x-3)$ and $y=\frac{1}{\sqrt{8}}(x-3)$.

9. $x=2+3t$, $y=5-4t$, the speed is $\frac{\sqrt{222}}{3}$.

10. The time it takes is $\sqrt{73}/3$ and the equations are $x=2+\frac{24}{\sqrt{73}}t$ and $y=-1+\frac{9}{\sqrt{73}}$.

11. In the graph, the horizontal does not intersect the lower part of the circle given by the middle equation so you solve $3=\frac{1}{5}x+4$ when $x<5$ giving $x=-5$ and $-2x+19=3$ when $x>7$ giving $x=8$.

12. $x=0$ or $x=-3$.

13. (a) $y=\frac{5}{4}-\frac{3}{4x}$ or $y=\frac{5x-3}{4x}$(b) $\alpha=\frac{3}{2\beta-5}-3$ or $\alpha=\frac{18-6\beta}{2\beta-5}$. (c) $x=\frac{7}{18}$.

14. The line equation is $y=2x+1$, the points of intersection are $(x,y)=\left(\frac{2\pm\sqrt{79}}{10},\frac{-6\pm\sqrt{79}}{10}\right)$ and the distance between them is $\sqrt{\frac{79}{5}}$.

15. $y=-5\left(x+\frac{3}{2}\right)^2+\frac{73}{4}$. It is a parabola which opens down with vertex at $\left(\frac{3}{2},\frac{73}{4}\right)$. The roots are $\frac{15\pm\sqrt{365}}{10}$.

16. The parabola opens down with vertex at $(0,1)$. The line has y-intercept at -6 and goes up (has positive slope. They should have two intersection points: $(-1+2\sqrt{2},-8+4\sqrt{2})$ and $(-1-2\sqrt{2},-8-4\sqrt{2})$.

17. 7.