Math 120 Midterm 2 Review

Part A - Mechanics

This section summarizes the math tools we talked about in Chapters 7-15. You may put some of the material in this part on your note sheet.

Parabolas and Quadratic Functions

The graph $y = f(x)$ of the quadratic function

$$f(x) = ax^2 + bx + c$$

with $a \neq 0$ is a parabola. The x-coordinate of the vertex is given by $-\frac{b}{2a}$. The y-coordinate is $f\left(-\frac{-b}{2a}\right)$. If $a > 0$, the parabola opens up and at $x = \frac{-b}{2a}$ the function $f(x) = ax^2 + bx + c$ has its minimum value $f(-b/2a)$. If $a < 0$ then the parabola opens down and at $x = \frac{-b}{2a}$ the function $f(x) = ax^2 + bx + c$ has its maximum value $f(-b/2a)$.

The values of x where $ax^2 + bx + c = 0$ are called the roots or zeros. They are the x intercepts of the graph of the function $y = f(x)$. A quadratic function may have two, one or no roots.

To solve the equation

$$ax^2 + bx + c = 0$$

we use the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Problems in this first section are for you to practice the mechanics so you can comfortably use them in longer problems. Make sure you can do all the exercises in this part before you attempt long problems. The midterm questions will not be like these. They will be more likely be like the questions from Part B below. But remember, you need to hit hundreds of balls for practice before you can face Federer on the court.

1. Where is the vertex of the parabola $y = -2x^2 + 4x + 13$? Does the function $f(x) = -2x^2 + 4x - 13$ have a maximum or minimum value? What is it? Does $f(x)$ have any roots? If yes, how many? Sketch a graph of the parabola with the information you have collected.

   Composition of Functions

   To form the composition $f(g(x))$, you take the output g(x) and put it into f. For example, if $f(x) = ax + b$ and $g(x) = e^x$, then $f(g(x)) = ae^x + b$ and $g(f(x)) = e^{ax+b}$. To compute the domain of $f(g(x))$, you make sure x is in the domain of g and $g(x)$ falls into the domain of f.

2. If $f(x) = 2 - x^2$ and $g(x) = \frac{1}{x-3}$, what are $f(g(x))$ and $g(f(x))$?

3. Let $f(x) = 2x + 1$. What are $f(f(x))$, $f(f(f(x)))$, $f(f(f(f(x))))$? Can you find a rule for the formula of f composed with itself n times?

   Remember the three rules for figuring out the domain of a function from its formula:

   (a) Do not divide by 0.

   (b) Do not take the square root of a negative number.

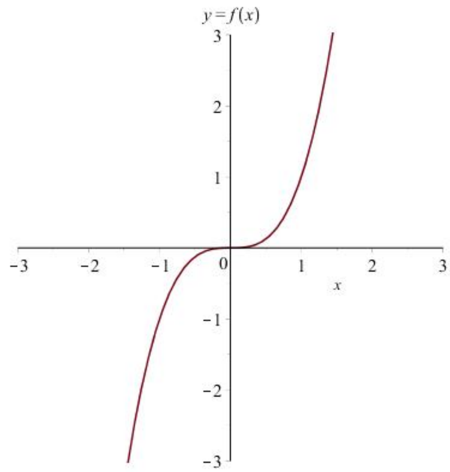
   (c) Do not take the logarithm of 0 or a negative number.

4. What is the domain of $f(x) = \frac{\ln(1-x^2)}{x}$?

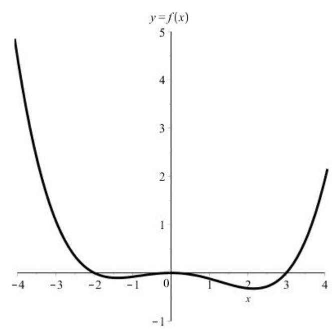
Inverse Functions

Two functions f and g are inverses of each other if they each undo what the other has done, i. e. $f(g(x)) = x$ and $g(f(x)) = x$. We write f^{-1} for g. (Or g^{-1} for f) in this case. To solve for the formula of $f^{-1}(x)$ from the fomrula of $f(x)$ you solve for y in the equation $f(y) = x$. If you think about it, since f^{-1} reverses what f does it takes y back to x so the roles of x and y are reversed. The idea that the roles of x and y are reversed is why the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are mirror images of each other about the mirror $y = x$: The point (a, b) is the mirror image of (b, a) with the mirror being $y = x$. Graph $y = x$, $(1, 3)$ and $(3, 1)$ to see this.

5. The graph of $y = f(x)$ is given. Graph $y = f^{-1}(x)$ on the same set of axes. Approximate $f^{-1}(2)$ from the graph.



6. Explain why the function with the following graph does not have an inverse.



The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}. Since it is much easier to compute the domains of functions from formulas (with the three rules above), if you compute the domains of f and f^{-1}, you know all four sets.

7. Compute the inverses of the following functions.

   (a) $f(x) = \frac{2x-3}{4x+7}$. Also find the domain and range of f.

   (b) $f(x) = \sqrt{x+1} + \sqrt{x-7}$.

   (c) $f(x) = \dfrac{1}{1 + \frac{1}{1+\frac{1}{x}}}$

Exponential and Logarithmic Functions

The function $f(x) = e^x$ has domain all numbers and range all positive numbers. Its inverse $f^{-1}(x) = \ln(x)$ has domain positive numbers and range all numbers. The inverse of $g(x) = b^x$ is $g^{-1}(x) = \log_b(x)$.

8. Solve for x is the following equations:

   (a) $2^x = 5$.

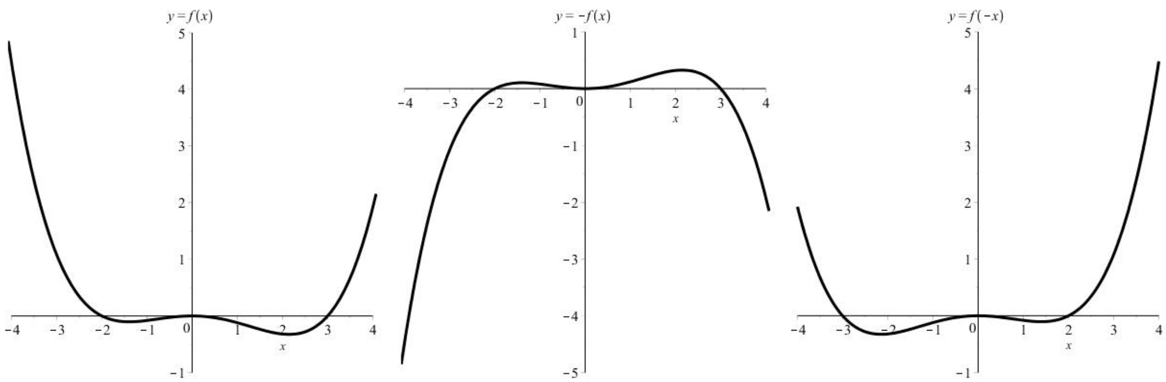
   (b) $2^{3^x} = 5$

   (c) $\ln(x^2) = 7$

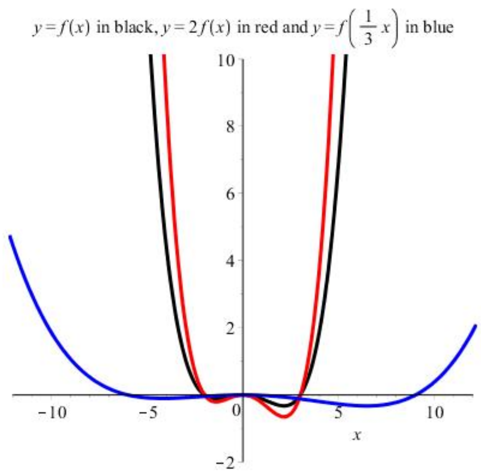
   (d) $\log_5 x = \log_2 3$

Graphing Functions Using Shifts, Stretches and Reflections

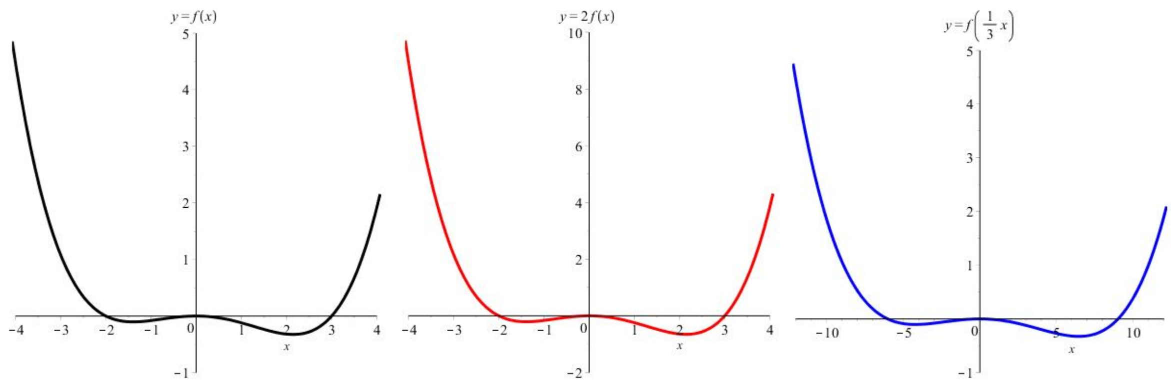
Given the graph of $y = f(x)$, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$about the x-axis (the line $y = 0$) and the graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$about the y-axis (the line $x = 0$).



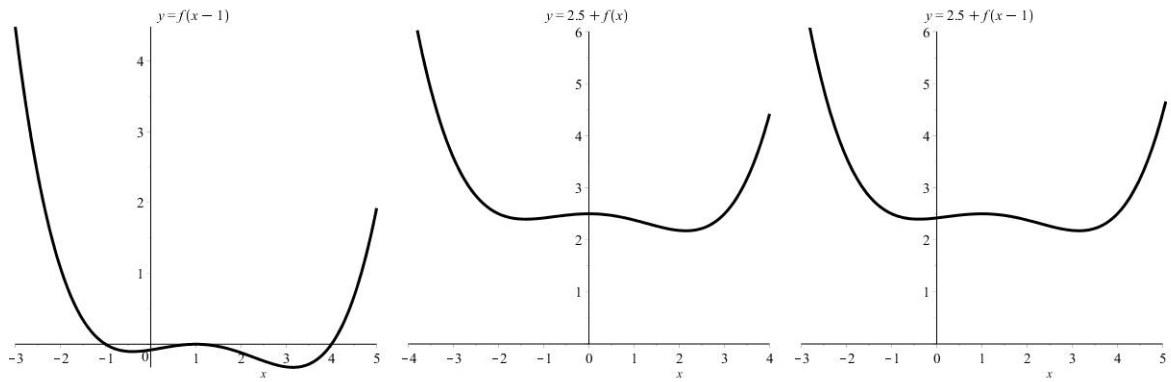
The graph of $y = f\left(\frac{x}{a}\right)$ with $a > 0$ is the graph of $y = f(x)$ stretched by a factor of a in the horizontal (x) direction. It is shrunk if $0 < a < 1$. The graph of $\frac{y}{a} = f(x)$ or $y = af(x)$ with $a > 0$ is the graph of $y = f(x)$ stretched by a factor of a in the vertical (y) direction. It is shrunk if $0 < a < 1$. In the examples below $a > 1$.



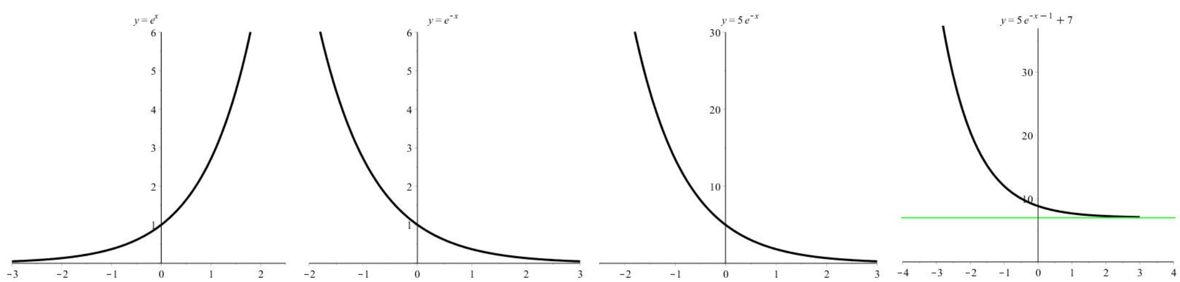
If we change the scaling, all three look the same.



The graph of $y-k=f(x-h)$ or $y=f(x-h)+k$ is the graph of $y=f(x)$ shifted h units right (left by $-h$ if $h<0$) and k units up (down by $-k$ if $k<0$)



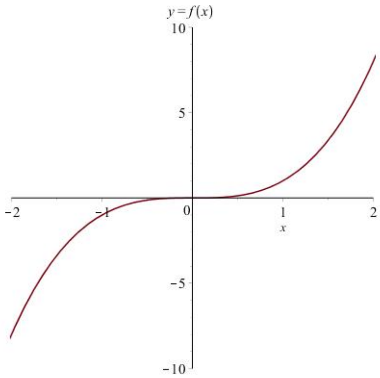
Several of these might be happening in a question. For example to graph $y=7+5e^{-x-5}$ you can graph $y=e^{x}$, $y=e^{-x}$, $y=5e^{-x}$ and then $y=5e^{-(x+5)}+7$ with $k=7$ and $h=-5$. Note the scaling change in the third graph for the vertical stretch and the shifted horizontal asymptote in green in the final graph.



9. Graph $y=-2(x-3)^2+7$ by first graphing $y=x^2$, $y=2x^2$ and $y=-2x^2$. You can do the first three on the same set of axes. Do the last one on a separate one.

10. Graph $y=3-\frac{2}{x+3}$ by first graphing $y=\frac{1}{x}$ and $y=\frac{2}{x}$ and $y=-\frac{2}{x}$. You can do the first three on the same set of axes. Do the last one on a separate one. Do not forget the asymptotes.

11. Below is the graph of $y=f(x)$. Graph $y=1-f(-2x)$ by first graphing $y=f(-2x)$ and $y=-f(-2x)$. The point $(1,1)$ is on the graph of $y=f(x)$. Mark the corresponding point on the graph of $y=1-f(-2x)$.



Linear to Linear Rational Functions

A linear to linear rational function is given by the quotient of two linear functions:

$$f(x) = \frac{Ax + B}{Cx + D}$$

with $C \neq 0$. We can divide the numerator and denominator by C

$$y = \frac{\frac{A}{C}x + \frac{B}{C}}{x + \frac{A}{C}}$$

and rename the constants so it looks like

$$f(x) = \frac{ax + b}{x + d}$$

with the coefficient of x in the denominator being 1. We always assume this in modeling questions so that we solve for the three unknowns a, b, d from the 3 pieces of information provided in the question.

12. Find a linear to linear rational function with $f(0) = 9$, $f(1) = 8$ and $f(3) = 7$.

To graph a linear to linear rational function, we use the fact that its graph is a reflection, dilation and shift of the familiar $y = \frac{1}{x}$. From

$$y = \frac{ax + b}{x + d}$$

we read the horizontal asymptote as $y = a$, the vertical asymptote as $x = -d$ and together with the y intercept of $f(0)$ we can graph the function.

13. Find the asymptotes and the intercepts of the function $y = \frac{2x-5}{3x+7}$ and use them to sketch a graph of the function.

14. Find a linear to linear rational function whose graph has y-intercept -3, x-intercept 2 and horizontal asymptote $y = 3$. What is the vertical asymptote?

Part B- Problems

The midterm questions will be like the problems in this section. You can find the old midterms and their solutions at http://www.math.washington.edu/∼m120/.The questions below are grouped into topics although one problem may be using several ideas. The questions below are all from second midterms.

Optimization Questions with Quadratic Functions

1. Autumn 14, Conroy, Question 2
2. Autumn 14, Ostroff, Question 1
3. Winter 14, Pezzoli, Question 4
4. Autumn 10, Conroy, Section B, Question 1

Problems with Compositions and Inverses of Functions, Algebra and Graphs

1. Winter 15, Ostroff, Question 2 (Graph the functions)
2. Autumn 14, Conroy, Question 4
3. Autumn 14, Ostroff, Question 3
4. Winter 14, Nichifor, Question 1
5. Autumn 13, Conroy, Question 4
6. Autumn 13, Ostroff, Question 2
7. Spring 13, Conroy, Question 4

Exponential Modeling

1. Winter 15, Ostroff, Question 4
2. Autumn 14, Conroy, Question 1
3. Autumn 14, Ostroff, Question 4
4. Winter 14, Nichifor, Question 4

Modeling with Linear to Linear Rational Functions

1. Winter 15, Ostroff, Question 1
2. Autumn 14, Conroy, Question 3
3. Autumn 14, Ostroff, Question 2
4. Spring 14, Ostroff, Question 4

Once you go through these, look at other old second midterms for more questions. Do at least one complete exam to test your timing skills. You are not responsible for Chapter 15 for this midterm. Do not solve problems involving angles or rotations.

Answers to Part A Problems

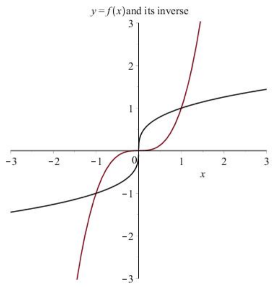
1. f has the maximum value 15 at $x = 1$. Its two roots are $\frac{4 \pm \sqrt{120}}{4}$.

2. $f(g(x)) = 2 - \dfrac{1}{(x-3)^2}$ and $g(f(x)) = -\frac{1}{x^2+1}$.

3. $f(f(x)) = 4x + 3$, $f(f(f(x))) = 8x + 6$. If you compose n times you get $2^n x + (2^n - 1)$.

4. $-1 < x < 1$ except for $x = 0$.

5. $f^{-1}(2) \approx 1.26$.



6. It is not one-to-one. For example, $f(-2) = f(0) = f(3) = 0$ so we cannot define $f^{-1}(0)$. Graphically, it does not pass the horizontal line test.

7. (a) $f^{-1}(x) = -\frac{7x+3}{4x-2}$. Domain of f is $x \neq -7/4$, the range of f is $y \neq 2$.

   (b) $f^{-1}(y) = \frac{64+12x^2+x^4}{4x^2}$.

   (c) $f^{-1}(x) = \dfrac{1}{\frac{1}{\frac{1}{x}-1} - 1}$ or $f^{-1}(x) = \frac{1-x}{2x-1}$.

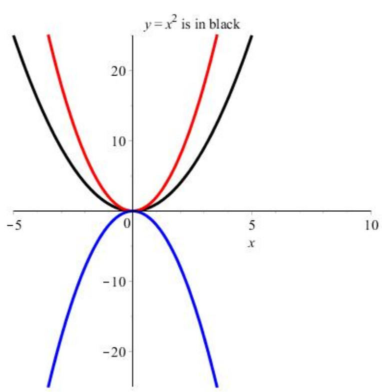
8. (a) $x = \log_2 5 = \frac{\ln 5}{\ln 2}$.

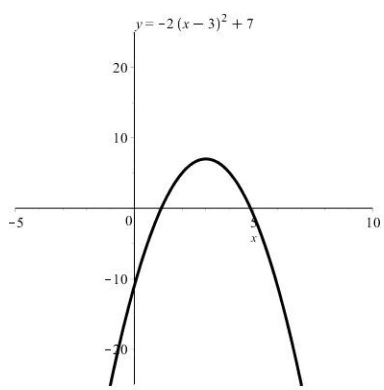
   (b) $x = \log_3(\log_2 5) = \dfrac{\ln\left(\frac{\ln 5}{\ln 2}\right)}{\ln 3}$

   (c) $x = e^{3.5}$.

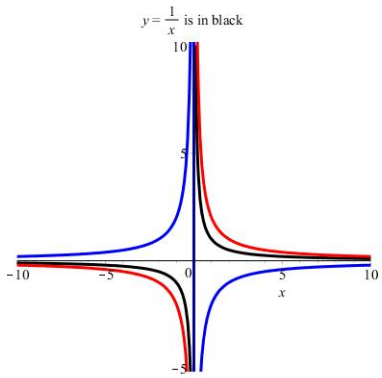
   (d) $x = e^{\frac{(\ln 3)(\ln 5)}{\ln 2}} = 3^{\frac{\ln 5}{\ln 2}} = 5^{\frac{(\ln 3)}{\ln 2}}$

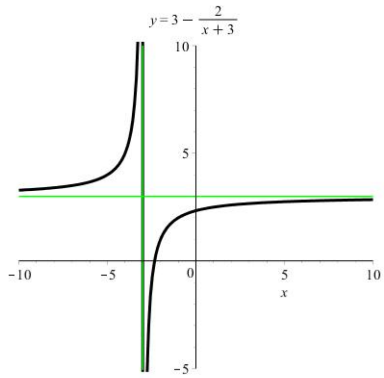
9. .



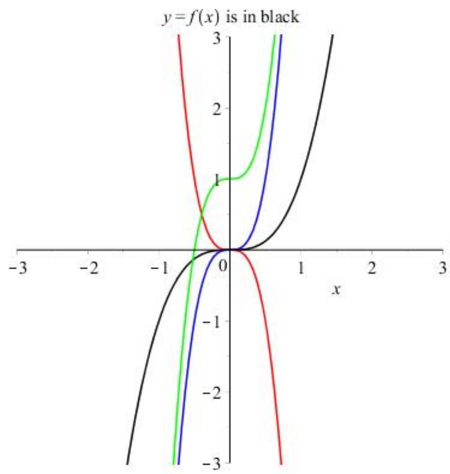


10. .



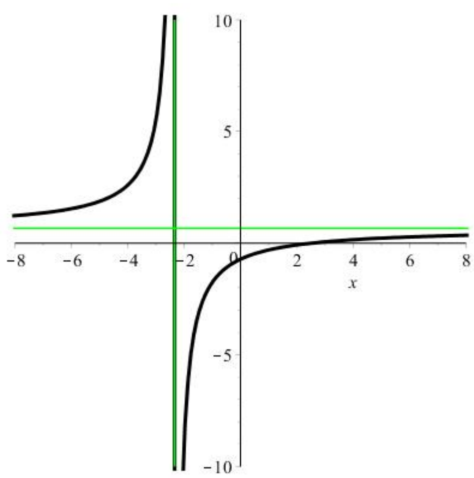


11. The answer is green.



12. $f(x) = \dfrac{5x+27}{x+3}$.

13. Horizontal asymptote $y = \frac{2}{3}$, vertical asymptote $x = -\frac{7}{3}$, y-intercept $-\frac{5}{7}$ and x-intercept $\frac{5}{2}$.



14. $f(x) = \dfrac{3x-6}{x+2}$.