Theorem: Let f be differentiable on I. Then, f' is increasing on I if and only if for any $a<b$ in I

$$f(x) < f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \quad \text{for } a<x<b.$$

Proof: $(\Rightarrow)$ Assume f is differentiable on I and that f' is increasing.

Let $a<b$, $a,b \in I$.

Let $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$

1 Then, $g(a)=g(b)=0$

2 By Rolle's Theorem, there is a c, $a<c<b$ such that $g'(c)=0$.

$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

3 + 4 Since f' is increasing, so is g'.

On (a,c), $g'(x) < g'(c) = 0$ so by Theorem 4.2.3 g decreases on $[a,c]$ so $0=g(a) > g(x)$ for any $a<x \le c$.

On (c,b), $g'(x) > g'(c)=0$ so by Thm 4.2.3 g increases on $[c,b]$ so $g(x) < g(b)=0$ for any $c \le x < b$.

Therefore, $g(x)<0$ for any x in (a,b).

($\Leftarrow$) Conversely, assume

$a<b$ implies $f(x) < f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$ for any $a<x<b$

① , Some algebra missing at (*)

Let $y \in I$

Let $a=y$, $x=y+k$, $b=y+h$ with $0<k<h<b-y$

then $$f(y+k) < f(y) + \frac{f(y+h)-f(y)}{h}(y+k-y)$$

or $$\frac{f(y+k)-f(y)}{k} < \frac{f(y+h)-f(y)}{h} \text{ when } 0<k<h<b-y$$

(1) so $$f'(y) = \lim_{k\to 0^+} \frac{f(y+k)-f(y)}{k} < \frac{f(y+h)-f(y)}{h} \text{ for any } 0<h<b-y$$

Similarly, let $a=y+h$, $x=y+k$, $b=y$ with $h<k<0$

then $$f(y+k) < f(y+h) + \frac{f(y)-f(y+h)}{-h}(y+k-y-h)$$

(*)

or $$\frac{f(y+k)-f(y)}{k} > \frac{f(y+h)-f(y)}{h} \quad h<k<0$$

(2) so $$f'(y) = \lim_{k\to 0^-} \frac{f(y+k)-f(y)}{k} > \frac{f(y+h)-f(y)}{h} \text{ for } h<0.$$

②

Now, given $y_1 < y_2$ in I

with $h = y_2 - y_1 > 0$, $y = y_1$ from (1) we get

$$f'(y_1) < \frac{f(y_2)-f(y_1)}{y_2-y_1}$$

with $h = y_1 - y_2 < 0$ $y = y_2$ from (2) we get

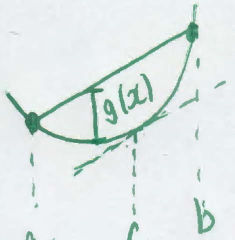
$$f'(y_2) > \frac{f(y_2)-f(y_1)}{y_2-y_1}$$

So, $f'(y_1) < f'(y_2)$ and f' is increasing.

Proof summary:

($\Rightarrow$) Assume f is differentiable on I and f' is increasing. Let $a<b$, $a,b \in I$

Let $g(x) = f(x) - \left[f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)\right]$



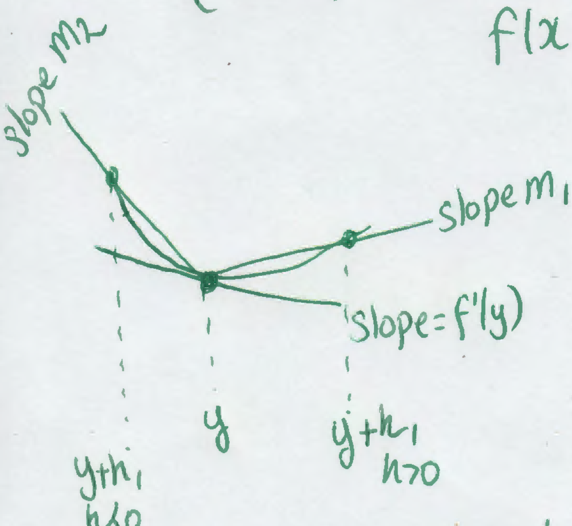
① $g(a) = g(b) = 0$

② $g'(c) = 0$ at some c, $a<c<b$.

③ g' is increasing so
$g'(x) < 0$ when $x<c$
$g'(x) > 0$ when $x>c$

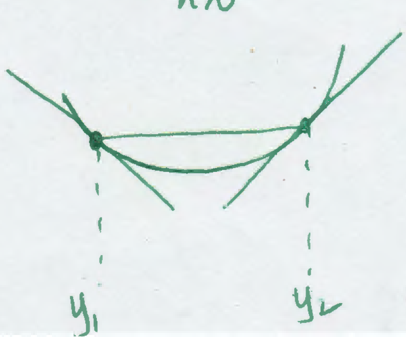
④ Use Thm 4.2.3 to show
$g(x) < 0$ for x in (a,b).
when $x \leq c$ and when $x \geq c$.

($\Leftarrow$) Assume $a<b$ implies
$$f(x) < f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \quad \text{for any } a<x<b$$



① $m_2 < f'(y) < m_1$, for any h as shown in picture

If $y_1 < y_2$,
② then $f'(y_1) < \dfrac{f(y_2)-f(y_1)}{y_2-y_1} < f'(y_2)$



So f' is increasing