Math 134 Final Review, Fall 2014

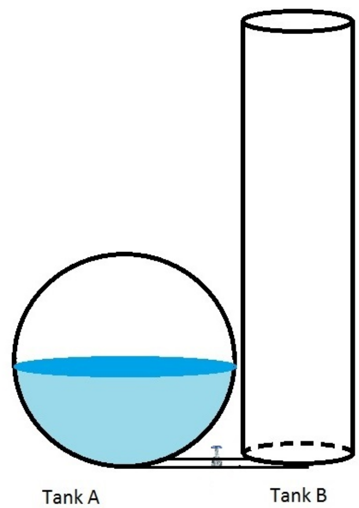
First, go over your homework problems. Also, look at at least one Math 124 and one Math 125 final exam from the archives to review the topics. In general, they are good sources of standard problems like differentiation drills, related rates, optimization, integration, volumes, areas, work, DE story problems. Besides proof by mathematical induction, and proof of continuity at a point using ϵ and δ, you should be able to use the theorems to do more general proof questions.

Here are some review questions. If you use a theorem in a question, make a note of which one.

1. Let f be increasing in the interval $[a,b]$. Let $A(x) = \int_a^x f(t)dt$. Prove that
$$A\left(\frac{x+y}{2}\right) \leq \frac{A(x)+A(y)}{2}.$$
Interpret this result geometrically.

2. The cross sections of a solid are squares perpendicular to the x-axis with their centers on the axis. If the square cut off at x has edge $2x^2$, find the volume of the resulting solid between $x=0$ and $x=a$.

3. Here is the question which almost made it to the second midterm:
Tank A is a sphere with radius R feet. Tank B is a cylinder with radius $R/2$ feet and height $4R$ feet. Both tanks have holes at their top to let air in. They are connected by a pipe whose volume is negligable. Initially, the valve between the tanks is closed and the spherical tank is half full. When the valve is opened, the height of the liquid in the two tanks will be the same by the basic law of fluid presure. Find the common height of fluid in the tanks.
When you have the equation for the common height h, you'll see why it was not on the midterm.



4. There is a function f, defined and continuous for all real x, which satisfies an equation of the form
$$\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C,$$
where C is a constant. Find the explicit fromula for $f(x)$ and find the constant C.

5. A boat sales parallel to a straight beach at a constant speed of 12 miles per hour, staying 4 miles offshore. How fast is it approaching a lighthouse on the shoreline at the instant it is exactly 5 miles from the lighthouse.

6. Show that
$$\int_0^x \frac{\sin t}{t+1} dt \geq 0 \text{ for all } x \geq 0.$$

7. The *Bernoulli Polynomials* are defined by induction as follows:
$$P_0(x) = 1 \qquad P_n'(x) = nP_{n-1}(x) \qquad \int_{)}^{1} P_n(x)dx = 0 \text{ if } n \geq 1.$$
   (a) Prove by induction that $P_n(x)$ is a ploynomial of degree n, the highest term being x_n.
   (b) Prove that $P_n(0) = P_n(1)$, if $n \geq 2$.

(c) Prove that $P_n(x+1) - P_n(x) = nx^{n-1}$, if $n \geq 1$.

8. Show that
$$\int_x^1 \frac{dt}{1+t^2} dt = \int_1^{1/x} \frac{dt}{1+t^2} dt, \text{ if } x > 0.$$

9. The *integral logarithm* is defined as
$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}, \text{ if } x \geq 2.$$
This function comes up in analytic number theory and it is a good approximation of the number of primes less than or equal to x. Show that $\mathrm{Li}(x) = \frac{x}{\ln x} + \int_2^x \frac{dt}{\ln^2 t} - \frac{2}{\ln 2}$.

10. A function f is *Lipschitz of order* α at x if there is a constant C such that
$$|f(x) - f(y)| \leq C|x-y|^\alpha$$
for all y in an interval of the form $(x-\beta, x+\beta)$, where β is a positive number. A function f is *Lipschitz of order* α in an interval if there is a constant C such that
$$|f(x) - f(y)| \leq C|x-y|^\alpha$$
for any x and y in the interval. Show that

    (a) If f is Lipschitz of order $\alpha > 0$ on (a,b), then f is uniformly continuous on that interval.

    (b) If f is differentiable at x, then f is Lipschitz of order 1 at x.

11. Prove that the function $f(x) = 2x^2 - x\sin x - \cos^2 x$ satisfies $f(x) = 0$ for precisely two numbers x.

12. Prove that $f(x) = \dfrac{x}{1-x^2}$ is one to one on $(-1,1)$ and find its inverse.

13. Find all functions satisfying $f'(t) = f(t) + \displaystyle\int_0^1 f(t)dt$.

14. Find all continuous functions which satisfy
$$(f(x))^2 = \int_a^x f(t)\frac{t}{1+t^2}dt.$$

15. Consider an object consisting of a spherical shell of mass 0.5 kilograms filled with 0.5 kilograms of sand. At time $t = 0$ seconds the object is dropped from an airplane at an initial speed of 0 meters per second. The shell has a hole in it so the sand begins to pour out at a constant rate of 1/10 kilogram per second as soon as the object is dropped.

    (a) Write down a formula for $m = m(t)$ the mass of the object after t seconds $(0 \leq t \leq 5)$.

    (b) Recall that the force on the object due to air resistance is given by $-kv$, where $k > 0$ is a constant. If we choose v to be positive when the object is falling, then v satisfies the differential equation
    $$m\frac{dv}{dt} + kv = mg\,,$$
    where g is the acceleration due to gravity. Find a formula for v in terms of t, k, and g for t between 0 and 5 seconds.

16. Suppose that $h : \mathbf{R} \to \mathbf{R}$ is a continuous function satisfying $h(x) > 2$ for all x. Let $f(x) = \displaystyle\int_1^x h(t)\,dt$.

    (a) Prove that f is invertible.

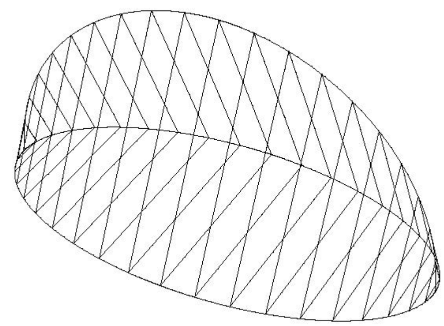
(b) Let $g = f^{-1}$. Prove that there is exactly one point x so that $g(x) = x$. (Prove that there is at least one such point, and prove that there is at most one such point.)

17. A bowl of irregular shape is being filled with water. Let h denote the height (in centimeters) of the water in the bowl, let $A(h)$ denote the surface area (in square centimeters) of the top surface of the water, and let $V(h)$ denote the volume of the water in the bowl (in cubic centimeters). It is known that $A(h)$ is a continuous function of h.

   (a) Find a relation between h, $A(h)$, and $V(h)$.

   (b) When $h = 10$ cm, water is flowing into the bowl at a rate of 5 cubic centimeters per minute. How fast is the height of the water in the bowl increasing at that time?

18. A tent has an elliptical base whose major and minor axes are 8 feet and 4 feet, respectively (So "$a = 4$ and $b = 2$.") The cross-sections of the tent perpendicular to the major axis of the tent are triangles. The top of the tent forms a semi-circle of radius 4 feet (see figure). What is the volume of the tent(in cubic feet)?



19. The current y in an AC electrical circuit with resistance 5 ohms, inductance 1 henry, voltage $\sin \omega t$ (in volts) satisfies the differential equation

$$\frac{dy}{dt} + 5y = \sin \omega t.$$

(The frequency ω is a positive constant.) Find the general solution y.