Math 134, Fall 2014, Midterm 1 Review

First, go over your homework problems. Look at the solutions for those problems that were not graded.

Computations

Evaluating limits, explicit diffferentiation, implicit differentiation, applications: tangent lines, first and second derivative tests, sketching curves, max min problems. This is mostly what you already knew coming into this class.

1. Look at the Review Problems at the end of the Chapters 1-3. We have not finished Chater 4 yet, so problems 1-42 on pages 232-233. Do not do them all. That is very time consuming.

2. Let $P(x) = x^3 + x + 1$. At which, if any, points on the graph of P does the tangent line pass through the origin?

3. Let $f(x) = \dfrac{9}{x} - \dfrac{9}{x^2}$. Find the critical points and inflection points of f, and draw a (reasonably careful) sketch of its graph.

4. Find the maximum possible area and the dimensions of a rectangle with sides parallel to the axes if its vertices lie on the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

5. Find the area and dimensions of the largest rectangle which has two vertices on the line $y = 2$ and two vertices on the portion of the hyperbola $y^2 - x^2 = 1$ where $1 \le y \le 2$.

6. Notice that the point $(x, y) = (0, 1)$ satisfies the equation
$$x^2 + \sin^2(x) + y^2 = 1\,.$$
One can show that this equation defines a twice differentiable function $y = f(x)$ for which $f(0) = 1$.
   (a) Use implicit differentiation to compute $f'(0)$.
   (b) Use implicit differentiation to compute $f''(0)$.
   (c) Using the results of parts (a) and (b), characterize the point $(0, 1)$. (For instance, it might be a local maximum for f, a local minimum, a point of inflection, or something else.) Explain your answer.

The Intermediate Value Theorem, Extreme Value Theorem, Rolle's Theorem and the Mean Value Theorem

1. Problems 23-38 in Section 4.1.

2. Problem 27 in Section 2.6.

3. Suppose that the function f is continuous for all $x \in R$ and that $f(0) = 0$. Suppose further that f is differentiable for $x \neq 0$ and that f' satisfies the inequality
$$|f'(x)| \le |x|$$
   (a) Use the Mean-Value Theorem to prove that f satisfies the inequality
$$|f(x)| \le x^2$$
   (b) Use the result of part (b) and the definition of the derivative as the limit of difference quotients to prove that f is differentiable at $x = 0$ and $f'(0) = 0$.

4. Suppose that f and g are differentiable functions such that $f'(x) = g(x)$ and $g'(x) = -f(x)$.

(a) Show that between any two consecutive zeroes of f there is exactly one zero of g, and between any two consecutive zeroes of g there is exactly one zero of f.

(b) Let $T(x) = (f(x))^2 + (g(x))^2$. Show that $T(x)$ must be constant. (Hint: how can you tell that a function is constant?)

(c) Now suppose that the equation $x = f(y)$ implicitly defines a differentiable function y. Compute y'; for full credit, write your answer in terms of x and constants – there should be no y in your answer. (You may assume that $x > 0$, $y > 0$, and $y' > 0$.)

Proofs

Proof by Induction. Proving a limit exists or does not exist - using ϵ and δ, pinching theorem, using the limit theorems, using the definition of the derivative, continuity

1. Give the definition of $\lim_{x \to a} f(x)$.
2. Problems 50-53 in Section 2.2.
3. Problem 62.
4. Let $f(x) = x^2 + x + 1$. Using only the definition of continuity (and limit theorems), prove that f is continuous at the origin.
5. Let $f(x) = \sqrt{x^2 + 9}$. Compute $f'(x)$ directly from the definition of derivative as a limit of difference quotients.
6. Suppose that f is continuous and $f(0) = 1$. Show, from the definition of continuity, that there is an interval I centered at 0 such that $f(x) > \frac{2}{3}$ for $x \in I$. (*Don't make this complicated; it's not.*)
7. Consider the function $f : R \to R$. Suppose that f satisfies the inequality

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in R$.

(a) Prove that f is continuous on R

(b) Give an example to show that f need not be differentiable on all of R.