How do you prove a limit does not exist?

In order to do do proofs, you do not need to know formal logic. After all, our brain uses logic all the time and we really know all we need to know. However, if you feel something is missing in your understanding of how we structure a proof because you have not studied formal logic before, here is a short summary of basic logic operations, their definitions and how we can formalize proof statements. As an example, we will do a proof of a function not having a limit at some point. Proving the limit does not exist is really proving that the opposite (negation) of the statement in the definition is true. What is the negation or opposite of a complicated statement like the one in the definition of limit? First, we start with basic logic.

I. Operations in Logic

A *statement* is something that is either true or false.

Examples:

1. 2=3.
2. The set of rational numbers is a subset of the set or real numbers.
3. For any $a \in \mathbf{Z}^+$, $\sqrt{a}$ ia an irrational number.
4. If f is differentiable at a, then it is continuous at a.

Which of the above statements are true?

We will use the letters P, Q and R for arbitrary statements. There are five basic logic operations. The logic operations NOT, AND, OR, IMPLIES and IFF are capitalized below to distinguish them from the English explanations. There are also symbols for these operations which are mentioned in a separate section at the end.

a. Negation - NOT

The negation of a statement P

$$\text{NOT } P$$

is true when P is false and it is false when P is true.

Example:
The negation of $x < 6$, (NOT $x < 6$), is $(x \geq 6)$. In other words, $x < 6$ is true when $x \geq 6$ is false and vice verse.

b. AND

The statement

$$P \text{ AND } Q$$

is true when *both* the statements are true. For example,

$$7 \text{ is prime AND } 7 \text{ is odd}$$

is a true statement. If one or both of the statements are false, the statement is false.
Examples:

1. 2 is prime AND 2 is odd.
2. 1 is prime AND 1 is odd.
3. 6 is prime AND 6 is odd.

All three are false statements.

In a proof, to prove a statement with AND you actually prove two pieces. For example, in order to prove $2 \leq x \leq 5$, which is short notation for

$$2 \leq x \text{ AND } x \leq 5,$$

you have to prove the two pieces. You can do that separately or together depending on the rest of the proof.

c. OR

The statement

$$P \text{ OR } Q$$

is true when at least one of them is true. For example, for any real number x, $x > 0$ OR $x \leq 0$.

Examples:

1. $2 = 3$ OR $1 = 1$.
2. $x \in \mathbf{Q}$ OR $x \notin \mathbf{Q}$.
3. $3 < 4$ OR $4 < 5$
4. $3 = 4$ OR $0 = 1$

All except the last one are true statements. In ordinary language, the word *or* sometimes has the hidden meaning that both of the things cannot be true at the same time. We understand *or* as *one or the other*. However, in logic, OR does not have that exclusive meaning. P OR Q is true when both are true.

d. IMPLIES

This is what we see in almost every statement of a proposition or theorem:

$$P \text{ IMPLIES } Q$$

or equivalently,

$$\text{IF } P, \text{ THEN } Q.$$

This statement is true in all cases *except* when P is true and Q is false. The statement "IF P, THEN Q" makes a claim about what would happen if P were true. So it holds if P is false. It is also true if P and Q are both true. The only time it fails is when P holds but Q does not. One way to think about "IF P, THEN Q" is to think of it as a promise. If you hold your end of the bargain, P, then I'll hold my end of the bargain, Q. You only complain if you do P but I do not do Q.

Examples:

1. I tell you that

   If you get 95% or higher on this course, I will give you a 4.0.

   The only time this will be false (when you come to my office and complain about your final grade) is if you have 95% or higher for your total and you see 3.9 or lower on your transcript. If you had a total of less than 95%, then I have not broken my promise if I gave you a 3.9. I have not broken my promise if I have given you a 4.0, either.

2. Here is a more mathematical example with a function f defined on the set of real numbers:

   If f is differentiable, then f is continuous.

   This is true. You cannot find a function which is differentiable (P holds) but not continuous (Q fails). The statement

   If f is continuous, then f is differentiable,

   is false. You would point to a counterexample, for example, $f(x) = |x|$ at $x = 0$.

In the wording of theorems, propositions, definitions etc., besides the phrases "P implies Q" and "If P, then Q" the statement P IMPLIES Q can also be expressed as "Q if P" or "P only if Q". The last one points that P cannot be true when Q is false.

e. IF AND ONLY IF

The statement

$$P \text{ IF AND ONLY IF } Q$$

is the short version of the the compound statement

$$(P \text{ IMPLIES } Q) \text{ AND } (Q \text{ IMPLIES } P).$$

When is this true? Well, if P and Q are both true, both implications would be true so the whole statement would be true. Also, if they are both false, then both statements would still be true: P being false makes P IMPLIES Q true, and Q being false makes Q IMPLIES P true. However, if one is true and the other is false, one of the implications would be false, making the compound statement "(P IMPLIES Q) AND (Q IMPLIES P)" false. So, P If AND ONLY IF Q is true when P and Q have the *same* truth values and false is they have *opposite* truth values. The IF AND ONLY IF is equality in logic. It shows that two statements are logically equivalent. We frequently write IFF in place of IF AND ONLY IF.

Proving an IFF statement is usually a two part proof. One part for the P IMPLIES Q part, another for the Q IMPLIES P part.

II. Negations of Compound Statements

One of the proof techniques we use is proof by contradiction. There, we assume that the result is false, that is, we assume that the opposite of the statement is true. So we need to find the negation of a statement. Another place where we would need the negation of a statement is if we are disproving a statement, which is really proving its negation. That is the example at the end. If the statement is simple like $x < 0$, we can find its negation $x \geq 0$ without much effort. But sometimes, the statement itself might be compound.

Example: We want to write a proof by contradiction of "If $x^2 < 9$, then $-3 < x < 3$. So we assume $x^2 < 9$, and, for a contradiction, we assume $-3 < x < 3$ is false. So we have to write the negation of "$-3 < x$ AND $x < 3$". When would the statement "$-3 < x$ AND $x < 3$" be false? When $x \leq -3$ OR when $x \geq 3$. So the negation of "$-3 < x$ AND $x < 3$" is $x \leq -3$ OR $x \geq 3$.

In general,

The negation of [P AND Q] is [(NOT P) OR (NOT Q)].

Similarly, the negation of an OR statement is an AND statement.

Example: If I tell you that I will grade your exams by Wednesday or bring cookies for class on Wednesday (to compensate for my laziness), you have a right to complain if I arrive Wednesday morning to class with the exams not graded and with no cookies.

Again, this is a general rule:

The negation of [P OR Q] is [(NOT P) AND (NOT Q)].

As an exercise, you can negate all the examples for OR and AND we have above. You can also do negations of messier compound statements.

Example: The negation of

$$P \text{ OR } (Q \text{ AND } R)$$

is

$$(\text{NOT } P) \text{ AND } [\text{NOT } (Q \text{ AND } R)]$$

which is

$$(\text{NOT } P) \text{ AND } [(\text{NOT } Q) \text{ OR } (\text{NOT } R)].$$

Finally, we negate an implication. Remember that P IMPLIES Q is false only when P is true but Q is false. So,

The negation of [P IMPLIES Q] is [P AND (NOT Q)],

P holds and Q fails, so the promise is broken.

III. Quantifiers and Negations of Statements with Quantifiers

Besides the basic logic operations, there are also two quantifiers. Is a certain statement true for every x or are there some x's for which it works. We have both of these in the definition of the limit. The first one

$$\text{for all } x \ P(x)$$

is true if the statement $P(x)$, which depends on x, holds for *all* x's, just like it says. It is called the *universal quantifier.*

Example: "For all x real, $x^2 \geq 0$," is a true statement whereas "For all x rational, $\sqrt{x}$ is rational," is a false statement.

To prove a for all statement, we start with "Let x be given and arbitrary" and try to prove the rest, the $P(x)$ part. This is exactly what we do in the limit proof: We let $\epsilon > 0$ (be arbitrary) and follow the rest of the proof to show that it works for any $\epsilon > 0$.

The second quantifier is the *existential quantifier*

$$\text{there exists an } x \text{ such that } P(x).$$

It is true when *there is at least one* x with the property $P(x)$. There may or may not be more x's with that property.

Example: "There exists a real x such that $x^2 = 5$" is a true statement whereas "There exists a rational x such that $x^2 = 5$" is a false statement.

To prove a there exists statement, you have to specifically show at least one that makes the statement true. If there are more than one, you do not have to list them all. So to prove the first statement in the example we would say yes, $\sqrt{5}^2 = 5$ or $x = \sqrt{5}$ works. This is another piece we have in our limit definition. We prove that there exists a δ with a certain property by explicitly writing it down in termes of the ϵ which comes before it in the proof.

What makes a for all statement false? For example, the statement,

$$\text{for all } x \text{ real } x^2 > 0$$

is false because it fails for at least (in this case exactly) one number, namely 0. To show that the statement is false, you would point out that there exists an x for which it fails. i.e.

$$\text{there exists a real } x, \text{ such that } x^2 \leq 0.$$

So, the negation of

$$\text{for all } x \ P(x)$$

is

$$\text{there exists an } x \text{ such that NOT } P(x).$$

Similarly, the negation of

$$\text{there exists an } x \text{ such that } P(x)$$

is

$$\text{for all } x \text{ NOT } P(x).$$

Example: The negation of the false statement

$$\text{there exists a rational } x \text{ such that } x^2 = 5$$

is

$$\text{for all } x \text{ rational } x^2 \neq 5$$

When we do a proof of a compound statement with quantifiers, we basically take the statement and make it the outline of our proof.

Example: In order to prove $\lim_{x \to 2}(3x + 1) = 7$ we take its statement

$$\text{for every } \epsilon > 0 \text{ there exists a } \delta > 0 \text{ such that } \ [|x - 2| < \delta \text{ implies } |(3x + 1) - 7| < \epsilon]$$

and write the outline of our proof in the same order as:

Let $\epsilon > 0$ be arbitrary (*this is a for all quantifier so it should work for any* ϵ)
Define $\delta = ...$ (*to show existance we have to exibit a certain* δ)
Assume $|x-2| < \delta$ (*to prove an implication you start with the first part and try to reach the second*)
.
.
.
Then, $|(3x+1)-7| < \epsilon$ (*finishing the proof of the implication*)

IV. Example: Proving the the Dirichlet function is not continuous anywhere.

Now, let's put most of these together to prove that the Dirichlet function

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous everywhere. First, let's prove that

$$\lim_{x\to 3} g(x) \neq g(3)$$

or

$$\lim_{x\to 3} g(x) \neq 1$$

since $g(3) = 1$. So, the negation of the statement

$$\lim_{x\to 3} g(x) = 1$$

holds. We will now negate the statement in the limit definition. Remember the limit statement:

$$\text{for every } \epsilon > 0 \text{ there exists a } \delta > 0 \text{ such that } [|x-3| < \delta \text{ implies } |g(x)-1| < \epsilon]$$

Now, we negate this statement step by step. The negation

$$\text{not } [\text{for every } \epsilon > 0 \text{ there exists a } \delta > 0 \text{ such that } [|x-3| < \delta \text{ implies } |g(x)-1| < \epsilon]]$$

is

$$\text{there exists an } \epsilon > 0 \text{ such that not } [\text{ there exists a } \delta > 0 \text{ such that } [|x-3| < \delta \text{ implies } |g(x)-1| < \epsilon]]$$

negating the for all part. Next comes the negation of the there exist part:

$$\text{there exists an } \epsilon > 0 \text{ such that for all } \delta > 0 \text{ not } [|x-3| < \delta \text{ implies } |g(x)-1| < \epsilon]$$

Note that, the properties $\epsilon > 0$ and $\delta > 0$ remain unchanged. Now, we have to negate the implication. Remember that the negation of "P implies Q" is "P and (not Q)". So we get

$$\text{there exists an } \epsilon > 0 \text{ such that for all } \delta > 0\, [|x-3| < \delta \text{ and not } [|g(x)-1| < \epsilon]]$$

Finally, the negation of $a < b$ is $a \geq b$ so we finally have

$$\text{there exists an } \epsilon > 0 \text{ such that for all } \delta > 0\, [|x-3| < \delta \text{ and } [|g(x)-1| \geq \epsilon]]$$

So, to prove that the limit is not equal to 1, we have to *find* the ϵ that makes this possibe *for all* δ and exibit *at least one* x with that property. Here is the complete proof of $\lim_{x\to 3} g(x) \neq 1$. As you read it, think about how we can come up with the ϵ and x which work.

Let $\epsilon = 1/2$.
Let $\delta > 0$ be arbitrary.

Choose an irrational number x such that $|x-3|<\delta$. For example, find an n such that $\pi \times 10^{-n} < \delta$ and let $x = 3 + \pi \times 10^{-n}$.

Then, $|x-3| = \pi \times 10^{-n} < \delta$, but since $g(x) = 0$, $|g(x) - 1| = |0-1| = 1 \geq 1/2$.

Therefore, $\lim_{x\to 3} g(x) \neq 1$.

As an exercise, prove that

$$\lim_{x\to\pi} g(x) \neq 0.$$

Now, you can generalize the above arguments to c rational or c irrational instead of 3 and π. The two proofs together would imply that $g(x)$ is not continuous anywhere on the real line.

V. Symbols

There are also symbols for the logic operators and the quantifiers. We do not use then in writing up proofs. You can use them in your drafts, if you like. We tend to use the implies symbol $\implies$, a lot in drafts or quick writing. The nice thing about using words instead of symbols is that words already have meanings so you know what they stand for. Here are the list of the symbols: The operations

$$\neg : \text{NOT}, \quad \wedge : \text{AND}, \quad \vee : \text{OR}, \quad \implies : \text{IMPLIES}, \quad \iff : \text{IFF}$$

and the quantifiers

$$\forall : \text{for all} \quad \exists : \text{there exists.}$$

For example, the steps of negating the limit definition would look like this. The limit definition is:

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \, [|x-3| < \delta \implies |g(x)-1| < \epsilon]$$

So the negation

$$\neg \left(\forall \epsilon > 0 \;\; \exists \delta > 0 \, [|x-3| < \delta \implies |g(x)-1| < \epsilon]\right)$$

is

$$\exists \epsilon > 0 \;\; \neg \left(\exists \delta > 0 \, [|x-3| < \delta \implies |g(x)-1| < \epsilon]\right)$$

which is

$$\exists \epsilon > 0 \;\; \forall \delta > 0 \neg [|x-3| < \delta \implies |g(x)-1| < \epsilon]$$

which is

$$\exists \epsilon > 0 \;\; \forall \delta > 0 \, [|x-3| < \delta \wedge \neg (|g(x)-1| < \epsilon)]$$

which finally becomes

$$\exists \epsilon > 0 \;\; \forall \delta > 0 \, [|x-3| < \delta \wedge |g(x)-1| \geq \epsilon]$$

Maybe, this was easier to follow, looks more like an equation.