Q: Let $x, y \geq 0$ be real numbers. Prove that

$$\left(\frac{x+y}{2}\right)^n \leq \frac{x^n+y^n}{2}, \text{ for every integer } n \geq 1.$$

Proof: By induction on n. Let $x, y \geq 0$.

When $n=1$,

$$\left(\frac{x+y}{2}\right)^1 = \frac{x+y}{2} = \frac{x^1+y^1}{2}.$$

So the formula holds when $n=1$.

Assume that the formula holds when $n=k$:

$$\left(\frac{x+y}{2}\right)^k \leq \frac{x^k+y^k}{2}.$$

First, for any two real numbers $x, y \geq 0$,

we have $x \geq y$ or $x < y$.

(*) [If $x \geq y$, then $x^k \geq y^k$ for $k > 0$ so

$(y^k - x^k)(y-x) = (-1)(x^k - y^k)(-1)(x-y) \geq 0$ since

$x^k - y^k \geq 0$ and $x - y \geq 0$.

If $x < y$, then $x^k < y^k$ for $k > 0$ so

$(y^k - x^k)(y-x) > 0$.

In any case, $(y^k - x^k)(y-x) \geq 0$ for any $x, y \geq 0$.

So, $0 \leq y^{k+1} - y^k x - x^k y + x^{k+1}$

or $y^k x + x^k y \leq x^{k+1} + y^{k+1}$

adding $x^{k+1} + y^{k+1}$ to both sides we get

$x^{k+1} + y^k x + y^{k+1} + x^k y \leq 2(x^{k+1} + y^{k+1})$

The left side factors to give

$$(x^k+y^k)(x+y) \leq 2(x^{k+1}+y^{k+1})$$

dividing by 4 we get

$$\left(\frac{x^k+y^k}{2}\right)\left(\frac{x+y}{2}\right) \leq \frac{x^{k+1}+y^{k+1}}{2}.$$

Now,

$$\left(\frac{x+y}{2}\right)^{k+1} = \left(\frac{x+y}{2}\right)^k\left(\frac{x+y}{2}\right)$$

$$\leq \left(\frac{x^k+y^k}{2}\right)\left(\frac{x+y}{2}\right) \text{ by the induction hypothesis}$$

$$\leq \frac{x^{k+1}+y^{k+1}}{2}, \text{ by the above computation.}$$

Therefore, the formula holds for $n=k+1$, finishing the induction and the proof.

(*) In the proof, we took this fact for granted:
If $x>y$ and $k>0$ then $x^k>y^k$.
It holds for any real number $k>0$, but how would you prove it?
If $k\in\mathbb{Z}^+$, you can use induction and the order properties of the real numbers (p.5 in textbook) to write a proof. This is all you need since in the proof k is a positive integer.
Another way to prove $(y^k-x^k)(y-x)\geq 0$ would be to factor y^k-x^k when $k\geq 2$ (when $k=1$, the product is a square) and use $x,y\geq 0$.