In class, we proved that if a continuous function is increasing on its interval, then its inverse is also increasing and continuous. Similarly, if a continuous function is decreasing on its interval, then its inverse is also decreasing and continuous. To finish the proof of Theorem 7.1.6, we need to prove the following:

Lemma If f is continuous and one-to-one on (a, b), then it is increasing on (a, b) or it is decreasing on (a, b).

Proof: Assume f is continuous and one-to-one on (a, b).

Let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$. Then since f is one-to-one, $f(x_1) \neq f(x_2)$. Now, we have two cases:

Case I: $f(x_1) < f(x_2)$. We'll prove in this case that f is increasing on (a, b). Let $s_1, s_2 \in (a, b)$ such that $s_1 < s_2$. Define two functions

$$g_1(t) = ts_1 + (1-t)x_1 \qquad \text{and} \qquad g_2(t) = ts_2 + (1-t)x_2,$$

where $0 \leq t \leq 1$. So both $g_1(t)$ and $g_2(t)$ are defined on the closed and bounded interval $[0, 1]$.

To complete the proof, do the following steps:

1. Show that $a < g_1(t) < b$ and $a < g_2(t) < b$ for any $t \in [0, 1]$.
2. Show that $g_1(t) < g_2(t)$ for any $t \in [0, 1]$.

   So now we have
   $$a < g_1(t) < g_2(t) < b, \text{ for any } t \in [0, 1].$$
   Define
   $$h(t) = f(g_2(t)) - f(g_1(t)), 0 \leq t \leq 1.$$
3. Show that $h(0) > 0$.
4. Show that $h(t) \neq 0$, for any $t \in [0, 1]$.
5. Show that $h(t) > 0$ for any $t \in [0, 1]$.
6. Prove that $f(s_1) < f(s_2)$.
   So, f is increasing. This concludes Case I.

   Case II: Assume $f(x_1) > f(x_2)$. Let $g(x) = -f(x)$.
7. Show that g is continuous and one-to-one and that $g(x_1) < g(x_2)$.

   Let $a < s_1 < s_2 < b$. We can apply the argument in Case I to $g(x)$ and conclude that $g(s_1) < g(s_2)$. Then, $f(x_1) > f(x_2)$ and f is decreasing, finishing Case II.

   Therefore, in any case, f is either increasing on (a, b) or decreasing on (a, b).