To prove **Theorem 5.3.1**, which implies **Lemma B.4.5** as we saw in class, we will first prove a **Lemma**. Then, you will use this in an induction to prove the Theorem as part of your homework.

Lemma Suppose f is continuous on $[a, b]$. Let P be a partition of $[a, b]$. Let $y \in [a, b]$ such that $y \notin P$. Then,

$$L_f(P) \leq L_f(P \cup \{y\}) \qquad \text{and} \qquad U_f(P \cup \{y\}) \leq L_f(P).$$

Proof: Let $P = \{x_0, x_1, ..., x_n\}$, m_i be the minimum value of f on $[x_{i-1}, x_i]$, M_i be the maximum value of f on $[x_{i-1}, x_i]$, and $\Delta x_i = x_i - x_{i-1}$. Then

$$L_f(P) = m_1\Delta x_1 + ... + m_n\Delta x_n \qquad \text{and} \qquad U_f(P) = M_1\Delta x_1 + ... + M_n\Delta x_n$$

Now, since $y \in [a, b]$ but $y \notin P$, $x_{i-1} < y < x_i$ for some $1 \leq i \leq n$. Then, let A be the maximum value of f on $[x_{i-1}, y]$, let B be the maximum value of f on $[y, x_i]$, let a be the minimum value of f on $[x_{i-1}, y]$ and let b be the minimum value of f on $[y, x_i]$. Clearly, $A \leq M_i$, $B \leq M_i$ being the maximums over smaller domains and $a \geq m_i$ and $b \geq m_i$, being the minimums over smaller domains. Now, the two lower sums $L_f(P)$ and $L_f(P \cup \{y\})$ only differ on the interval which contains y so

$$L_f(P\cup\{y\})-L_f(P) = a(y-x_{i-1})+b(x_i-y)-m_i\Delta x_i \geq m_i(y-x_{i-1})+m_i(x_i-y)-m_i\Delta x_i = m_i(x_i-x_{i-1}-\Delta x_i) = 0$$

so

$$L_f(P \cup \{y\}) \geq L_f(P)$$

and the two upper sums $U_f(P)$ and $U_f(P \cup \{y\})$ also only differ on the interval which contains y so

$$U_f(P\cup\{y\})-U_f(P) = A(y-x_{i-1})+B(x_i-y)-M_i\Delta x_i \leq M_i(y-x_{i-1})+M_i(x_i-y)-m_i\Delta x_i = M_i(x_i-x_{i-1}-\Delta x_i) = 0$$

so

$$U_f(P \cup \{y\}) \leq U_f(P).$$