The following is a proof of **Theorem 7.1.8** or **Theorem B.3.2**. It is essentially the same as the one in the book, with some minor variable name changes: I used y in place of t, c in place of a and e in place of b. The first one is because $y = f(x)$ is standard, the last two are because I have my interval as (a, b) using up those letters. The main change is that I have more steps which are numbered so you can write the reasons referring back to previous steps, if necessary.

Theorem 7.1.8 Let f be a on-to-one function differentiable on (a, b). Let $c \in (a, b)$, $f(c) = e$ and $f'(c) \neq 0$. Then, f^{-1} is differentiable at e and

$$\left(f^{-1}\right)'(e) = \frac{1}{f'(c)}.$$

Proof: Let f be a on-to-one function differentiable on (a, b). Let $c \in (a, b)$, $f(c) = e$ and $f'(c) \neq 0$. To prove

$$\left(f^{-1}\right)'(e) = \frac{1}{f'(c)}$$

we'll show that

$$\lim_{y \to e} \frac{f^{-1}(y) - f^{-1}(e)}{y - e} = \frac{1}{f'(c)}.$$

Let $\epsilon > 0$ be given. First, the function $g(z) = \frac{1}{z}$ is continous at every $z \neq 0$, in particular at $z = f'(c)$, so there exists a $\delta_1 > 0$ such that

$$0 < |z - f'(c)| < \delta_1 \qquad \text{implies} \qquad \left| \frac{1}{z} - \frac{1}{f'(c)} \right| < \epsilon. \tag{1}$$

Since f is differentiable at c,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

so using the definition of the limit with $\epsilon = \delta_1$, there is a $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \qquad \text{implies} \qquad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \delta_1. \tag{2}$$

Now, let $x = f^{-1}(y)$ and $c = f^{-1}(e)$. Since f^{-1} is continuous at e, using $\epsilon = \delta_2$ in the definiton of continuity, there exists a $\delta_3 > 0$ such that

$$0 < |y - e| < \delta_3 \qquad \text{implies} \qquad \left| f^{-1}(y) - f^{-1}(e) \right| < \delta_2. \tag{3}$$

Now, let $\delta = \delta_3$.

For the rest of the steps, give a reason, completing computations if necessary:

1. Assume $|y - e| < \delta$.

2. Then, $|x - c| < \delta_2$

3. So
$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \delta_1.$$

4. Then,
$$\left| \frac{1}{\frac{f(x)-f(c)}{x-c}} - \frac{1}{f'(c)} \right| < \epsilon.$$

5. So,
$$\left| \frac{f^{-1}(y) - f^{-1}(e)}{y - e} - \frac{1}{f'(c)} \right| < \epsilon.$$

6. Therefore,

$$0 < |y - e| < \delta \qquad \text{implies} \qquad \left| \frac{f^{-1}(y) - f^{-1}(e)}{y - e} - \frac{1}{f'(c)} \right| < \epsilon.$$

7. So

$$\lim_{y \to e} \frac{f^{-1}(y) - f^{-1}(e)}{y - e} = \frac{1}{f'(c)}.$$

and

8.

$$\left(f^{-1}\right)'(e) = \frac{1}{f'(c)}$$