Math 135, Winter 2015, Midterm 1 Review Questions

1. Let S be the set of numbers of the form $(-1)^{n^2} \dfrac{n + \frac{1}{n!}}{n+1}$ for $n \geq 2$. Explain why S does or does not have a least upper bound. If it has a least upper bound, what is it? Answer the same question about greatest lower bounds.

2. Let $\{a_n\}$ be the sequence defined inductively by $a_1 = 1$, $a_{n+1} = \dfrac{1}{a_n^4 + 16}$.

   (a) Show that $\{a_n\}$ is a Cauchy sequence.

   (b) Show that $\{a_n\}$ converges to a solution of the equation $x^5 + 16x - 1 = 0$.

   (c) Show that if $\{b_n\}$ is the sequence defined by $b_1 = 2$, $b_{n+1} = \dfrac{1}{b_n^4 + 16}$, then $\{b_n\}$ is convergent and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

3. Show that the equation
$$x = 1 + \int_0^x \frac{\cos(t)}{t^2 + 4}\, dt$$
has one and only one solution.

4. (a) Consider the sequence $\{a_k\}$ defined by $a_0 = 1$, $a_{n+1} = \pi + \frac{1}{2}\sin(a_n)$. Does the series converge? Why or why not?

   (b) Consider the series $\{b_k\}$ defined by $b_0 = 0$, $b_{n+1} = \pi + \frac{1}{2}\sin(b_n)$. Find $\lim_{n\to\infty} b_n$. (Hint: Compute b_1 and b_2.)

   (c) What is the relation between $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$?

5. For each integer $n \geq 1$, let $a_n = 2\ln(3n-1) - \ln(2n^2 + 2n + 3)$. Does the sequence $\{a_n\}$ converge or diverge? If it converges, what is the limit?

6. Consider the sequence $\{a_k\}$ defined as follows:
$$a_0 = 1 \qquad a_{n+1} = 1 - a_n/2 \text{ for } n = 0, 1, 2, \ldots$$
Show that the sequence converges to 2/3.

7. Problem 47 in Section 12.3.

8. For which real numbers α and β does the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^\alpha (\ln k)^\beta}$ converge?

9. Test the following series for convergence, and (where appropriate) test for absolute and conditional convergence.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{k^k}{k!}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^k \frac{k+2}{k^2+k}$

   (d) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{k+2}{k^3+k}$

(e) $\sum_{k=0}^{\infty} \frac{k\sin(k\pi)}{k^2+1}$

(f) $\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k^3+\ln k+1}$

(g) $1 + \frac{1\cdot 2}{1\cdot 3} + \frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \dots$

10. Give the Taylor series about 0 of the function $f(x) = \int_0^x \sin(t^2)\,dt$. For what values of x does the series converge to $f(x)$? Justify your answer.

11. Find the first three non-zero terms in the series expansion of $\arcsin(x)$ about $x = 0$.

12. Let $f(x) = e^{-1/x^2}$, if $x \neq 0$ and let $f(0) = 0$.

    (a) Prove that $\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = 0$ for any integer $n \geq 0$.

    (b) Prove that for $x \neq 0$ and $k \geq 1$, the kth derivative $f^{(k)}(x) = \frac{e^{-1/x^2}}{x^{3k}} P_k(x)$, where $P_k(x)$ is a polynomial of degree at most $2k-2$.

    (c) Show that $f^{(k)}(0) = 0$ for all $k \geq 1$. What does it say about the Taylor Series for $f(x)$ near 0? Where does it converge? What does it converge to?

13. What is an upper bound for the error in estimating $\sin(x)$ with its Taylor polynomial of degree 5 near $x = 0$ if x is in the interval $[-\pi, \pi]$

14. Estimate $\sqrt{10}$ to 4 decimal places using a Taylor polynomial.

15. Find an interval J centered at 0 so that the error upper bound in approximating $\ln(1+x)$ with its third degree Taylor polynomial near $x = 0$ is at most 0.0001 for any x in J.

16. Consider the function f defined on the open interval $(-1, 1)$ by the formula

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k^2+1}, \quad |x| < 1.$$

    Notice that $f(x) = \sum_{k=0}^{n} \frac{x^k}{k^2+1} + R_n(x)$, where $R_n(x) = \sum_{k=n+1}^{\infty} \frac{x^k}{k^2+1}$.

    Show that $R_n(x)$ satisfies the inequality

$$|R_n(x)| \leq \frac{1}{(n+1)^2+1} \cdot \frac{|x|^{n+1}}{1-|x|}.$$

17. For each of the following power series: (i) find the radius of convergence, (ii) find the interval of convergence, (iii) determine the values of x for which the series is absolutely convergent and the values of x for which the series is conditionally convergent.

    (a) $\sum_{k=1}^{\infty} \frac{2^k \ln(k+1)}{k} x^k$.

    (b) $\sum_{k=1}^{\infty} \ln(k+1) \frac{x^k}{e^k}$.

    (c) $\sum_{k=0}^{\infty} \frac{2^k x^k}{k+1}$.

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{2^k(k+1)^2}$

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k \ln(k+1)}{k} x^k$

18. Show that for any real number c,

$$\lim_{x\to\infty} \left(\frac{x+c}{x-c}\right)^x = e^{2c}\,.$$

19. Evaluate the following limits in any way you wish.

(a) $\displaystyle\lim_{x\to 0} \frac{x\sin(x^2) - \sin(x^3)}{\sin(x^7)}$

(b) $\displaystyle\lim_{x\to 0} \frac{\cosh x - \cos x}{\sin x^2}$

(c) $\displaystyle\lim_{x\to 0} \frac{\cosh x - \cos x}{x^2}$

(d) $\displaystyle\lim_{x\to 0} \frac{\cos x - \cos 2x}{x \sin 4x}$.

(e) $\displaystyle\lim_{x\to 0} \frac{x^2\cos(x^2) - \sin(x^2)}{\sin(x^6)}$

(f) $\displaystyle\lim_{x\to 0} \frac{\sin(x^8) - x^6\cos(x^2)}{\sin(x^4) - x^4}$

20. Evaluate the integrals or explain why you can't.

(a) $\displaystyle\int_0^{\infty} \frac{\cos(x)}{(2+\sin(x))^2}\,dx$

(b) $\displaystyle\int_{-1}^{1} \frac{x}{1-x^2}\,dx$

(c) $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{(1+x^6)^2}\,dx$

(d) $\displaystyle\int_0^{1} \frac{dx}{x^{2/5}}$

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$