Math 135, Winter 2015, Midterm 2 Review

Topics

- Picard iteration
- linear degree n equations, linear independence, form of the general solution in both the homogeneous and non-homogeneous cases
- solving linear constant coefficient homogeneous equations using the characteristic equation
- finding a second solution to a linear homogeneous equation using reduction of order
- solving linear constant coefficient nonhomogeneous equations using the method of undetermined coefficients (and, secondarily, using variation of parameters)
- Laplace transforms and inverse Laplace transforms: basic formulas, using tables, discontinuous forcing functions
- solving linear constant coefficient equations using Laplace transforms
- power series solutions

Formulas you will have on the exam

Besides the Laplace Table sheet you used on the quiz, you will have the following on your exam:

Picard iteration for $y' = f(t, y)$, $y(t_0) = y_0$

$$y_n = y_0 + \int_{t_0}^{t} f(u, y_{n-1})du$$

Reduction of order: If y_1 is a solution of the homogeneous second order linear differential equation $y'' + p(t)y' + q(t)y = 0$, then the second solution is given by $y_2(t) = y_1(t)v(t)$ where $u(t) = v'(t)$ and

$$u(t) = e^{-\int \frac{2y_1' + py_1}{y_1} dt}.$$

Variation of Parameters: If y_1 and y_2 are two independent solutions of the second order homogeneous linear differential equation with constant coefficients $y'' + by' + cy = 0$, then the particular solution of $y'' + by' + cy = f(t)$ is given by $y_p = u_1y_1 + u_2y_2$ where

$$u_1' = -\frac{y_2(t)f(t)}{W(y_1, y_2)} \quad \text{and} \quad u_2' = \frac{y_1(t)f(t)}{W(y_1, y_2)}$$

and $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the Wronskian.

Practice problems

1. Is there a differential equation of the form $y'' + p(t)y' + q(t)y = 0$, with p and q continuous on the open interval $-1 < t < 1$, for which the functions $y = t^4$ and $y = t^6$ are both solutions? Explain your answer.

2. The solution of the initial value problem

$$y'' + (t + 1)y' + y = 0\,, \quad y(0) = 1 \quad y'(0) = 0$$

can be expressed in the form $y = \sum_{k=0}^{\infty} a_k t^k$. Find the recursion formula for the coefficients.

3. Solve the initial value problem

$$y'' - 2y' + y = f(t)\,, \quad y(0) = 0\,, \quad y'(0) = 1\,.$$

where $f(t) = \begin{cases} 4, & 0 \le t < 3, \\ 7, & 3 \le t. \end{cases}$

4. Let c be a constant, and let $y(t)$ be the solution to the initial value problem

$$y'' - 4y = f(t), \quad y(0) = 0,\ y'(0) = 0.$$

where $f(t) = \begin{cases} 0, & 0 \le t < 1, \\ 1, & 1 \le< 3. \\ c+1, & 3 \le t. \end{cases}$

For what value(s) of c does $y(t)$ satisfy $\lim_{t\to\infty} y(t) = 0$?

5. The solution to the initial value problem

$$(1+t^2)y'' + y' - 2y = 0, \quad y(0) = 1,\ y'(0) = 2,$$

can be expressed in the form $y = \sum_{n=0}^{\infty} a_n t^n$. Find the recursion formula for the coefficients. Also find a_0, a_1, a_2 and a_3.

6. Find the general solution to the equation

$$y'' + 2y' + 2y = 2\cos t.$$

7. Let $f(t)$ be a continuous function, and suppose that $y_1(t)$ and $y_2(t)$ are solutions to the equation

$$y'' + 4y = f(t)$$

satisfying the initial conditions

$$y_1(0) = 2,\ y_1'(0) = 2, \quad y_2(0) = 2,\ y_2'(0) = 0.$$

Find $y_1(t) - y_2(t)$.

8. The function $y_1 = t$ is one solution to the differential equation

$$y'' + \frac{2}{t}y' - \frac{2}{t^2}y = 0.$$

Solve the initial value problem

$$y'' + \frac{2}{t}y' - \frac{2}{t^2}y = 0, \quad y(1) = 0,\ y'(1) = 6.$$

9. Find the general solution of the differential equation

$$y'' + y = 2t^2 + t\,.$$

10. Find a particular solution of the differential equation $y'' - y = (1+t^2)^{-1}$. Express your answer in terms of integrals, and do not attempt to evaluate these integrals.

11. Is there a differential equation of the form

$$y'' + p(t)y' + q(t)y = 0\,,$$

with p and q continuous on the open interval $-1 < t < 1$, for which the functions $y = t^4$ and $y = t^6$ are both solutions? Explain your answer.

12. What is the general solution of the diffential equation

$$y'' - 2y' + y = e^t\,?$$

13. Solve the initial value problem $y''+\omega_0^2 y = e^t - e^{-t}$, $y(0) = 0$, $y'(0) = 0$ using the method of undetermined coefficients. (Here ω_0 is a positive constant.)

14. Consider the following differential equation:

$$y'' - 2ty' + 2\lambda y = 0\,,$$

where λ is a constant. (This is called *Hermite's diffential equation* and it arises in the solution of the "Quantum Harmonic Oscillator")

(a) Find the recursion formula for the coefficients of the power series solution $y(t) = \sum_{k=0}^{\infty} a_k t^k$.

(b) Suppose that the initial conditions are $y(0) = 1$, $y'(0) = 0$ and that $\lambda = 2n$ for n a positive integer. Using the recursion formula you found in part (a), show that the solution is a polynomial. (The solution is called a "Hermite Polynomial".

15. Solve the initial value problem

$$t^2y'' + ty' - y = 0\,, \quad t > 0\,, \quad y(1) = 1\,, \quad y'(1) = 0$$

Hint: What happens to the left hand side of the differential euation if $y(t) = t^r$.

16. Suppose that $Y(s) = \dfrac{4}{(s-1)(s+1)(s^2+1)}$ is the Laplace transform of the solution of the initial value problem

$$y'' - y = f(t)\,, \quad y(0) = 0, \quad y'(0) = 0\,.$$

(a) What is $f(t)$? (b) What is the solution?

Application Problems

1.