Summary of the 2nd Order Constant coefficient ODE examples

This is a summary of pages 313-368 in TP. After each case, example numbers are given to use with the Driven Damped Oscillator from the Wolfram Demonstrations Project available at
http://demonstrations.wolfram.com/DrivenDampedOscillator/.
First, you have to download the Wolfram CDF player. There is a link on the same page. The solutions have the initial condition $x'(0) = 0$. Vary the numbers in the examples to see what happens.

I. Unforced (homogeneous) case $mx'' + cx' + kx = 0$

A. Undamped $c = 0$

This is *simple harmonic motion* with the solution $x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$ where $\omega_0 = \sqrt{k/m}$ is the natural frequency. The equation can also be written in the form

$$x(t) = A\cos(\omega_0 t - \theta)$$

where $A = \sqrt{C_1^2 + C_2^2}$ is the amplitude of the motion and $\tan\theta = C_2/C_1$.

Example: Try mass $m = 4$, spring constant $k = 1$, damping friction $c = 0$, initial position $x(0) = 1$ in the Driven Damped Oscillator. The forcing amplitude must be 0 to make $f(t) = 0$.

B. Damped $c > 0$

1. It is **overdamped** if $c^2 - 4mk > 0$. There are two negative real roots.The solution is

   $$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

   and $x(t) \to 0$ as $t \to \infty$.

2. It is **critically damped** if $c^2 - 4mk = 0$. There is one real negative root.The solution is

   $$x(t) = C_1 e^{rt} + C_2 t e^{rt}$$

   and $x(t) \to 0$ as $t \to \infty$.

3. It is **underdamped** if $c^2 - 4mk = 0$. There are two complex roots $\alpha \pm \omega_0 i$ with $\alpha < 0$ and $\beta < \omega_0$.The solution is$x(t) = e^{\alpha t}\left(C_1 \cos\beta t + C_2 \sin\beta t\right)$ or

   $$x(t) = e^{\alpha t} A\cos(\beta t - \theta)$$

   The solution $x(t) \to 0$ as $t \to \infty$ and keeps oscillating.

   Examples: Try mass $m = 1$, spring constant $k = 1$, and vary the damping friction $c = 4.2$, $c = 4$ and $c = 0.2$ in the Driven Damped Oscillator. The forcing amplitude must be 0 to make $f(t) = 0$.

II. Forced (nonhomogeneous) case $mx'' + cx' + kx = A_0\cos(\omega t)$

A. Undamped $c = 0$

1. $\omega \neq \omega_0$. The solution is of the form

   $$x(t) = A\cos(\omega_0 t - \theta) + \frac{A_0/m}{\omega_0^2 - \omega^2}\cos(\omega t)$$

   If $x(0) = x'(0) = 0$ we can reorganize x to look like

   $$x(t) = \frac{A_0/m}{\omega_0^2 - \omega^2}\sin\left(\frac{\omega_0 - \omega}{2}t\right)\sin\left(\frac{\omega_0 + \omega}{2}t\right)$$

   and with $\omega_0 - \omega$ small compared to $\omega_0 + \omega$, we get one sine oscillation inside another.

   Example: Try $k = 4$, $c = 1$, $m = 0.25$, and the forcing frequency $\omega = 7$. I get a nice picture ($x(t)$ is the black curve) when the time interval is up to 88.

2. $\omega = \omega_0$. This is the **resonance** case. The solution is

$$x(t) = A\cos(\omega_0 t - \theta) + \frac{A_0}{2m\omega_0} t \sin(\omega_0 t).$$

The amplitude will increase with time.

Example: I can't get an example to work in this case. There might be a bug in the code.

B. Damped $c > 0$

First, note that when $c > 0$, the homogeneous solution $y_h(t) \to 0$. So, eventually, $y(t) \approx y_p(t)$. The particular solution is

$$y_p(t) = \frac{A_0}{m\Delta} \cos(\omega t - \theta)$$

where $\Delta = \sqrt{(\omega_0^2 - \omega^2)^2 + (\omega c/m)^2}$. The amplitude $A_0/m\Delta$ is maximized when $\omega = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}}$. This is called the **resonant frequency**. The corresponding maximum amplitude is $\frac{A_0}{c\sqrt{\omega_0^2 - (c/2m)^2}}$.

Example: You can set $k = 9$, $c = 2$, $m = 1$ and the forcing frequency $\omega = \sqrt{7} \approx 2.646$. Observe that the amplitude of the motion is large compared with the amplitude of the forcing function.