Consider the closed interval $I$. Let $\mathcal{C}$ denote the set of all continuous functions on the interval $I$. For $Y \in \mathcal{C}$, set

$$\|Y\| = \max_{t \in I} |Y(t)|$$

For $Y_1, Y_2 \in \mathcal{C}$, we call $\|Y_1 - Y_2\|$ the *distance* between $Y_1$ and $Y_2$. Notice that

(1) $|Y_1(t) - Y_2(t)| \le \|Y_2 - Y_1\|$

(2) $\|Y_1 - Y_2\| = 0$ implies $Y_1 = Y_2$.

(3) The triangle inequality holds:
$$\|Y_1 + Y_2\| \le \|Y_1\| + \|Y_2\| .$$

A sequence $\{Y_n\}$ of functions in $\mathcal{C}$ is said to *converge uniformly* to a function $Y$ if for every $\epsilon > 0$, there is an integer $N$ such that $\|Y_n - Y\| < \epsilon$ for all $n \ge N$.

A sequence $\{Y_n\}$ of continuous functions in $\mathcal{C}$ is said to be a *Cauchy sequence (of continuous functions)* if for every $\epsilon > 0$, there is an integer $N$ such that $\|Y_n - Y_m\| < \epsilon$ for all $n, m \ge N$.

**Exercise.** *Convince yourself that if $\{Y_n\}$ is a Cauchy sequence in $\mathcal{C}$, then for all $t \in I$, the sequence $\{Y_n(t)\}$ is a Cauchy sequence of numbers, and therefore a convergent sequence.*

Suppose that $\{Y_n\}$ is a Cauchy sequence of functions in $\mathcal{C}$. Define the limit of the sequence to be the function $Y : I \to \mathbb{R}$ defined by

$$Y(t) = \lim_{n \to \infty} Y_n(t) . \qquad (*)$$

(By the exercise above, this limit exists.) We will also write $Y = \lim_{n \to \infty} Y_n$.

**Theorem 1.** *Let $\{Y_n\}$ a Cauchy sequence of functions, and let $Y$ be the function defined in Equation* (*). *Then $Y$ is continuous and the sequence $\{Y_n\}$ converges uniformly to $Y$.*

**Proof.** Choose any $\epsilon > 0$. Because $\{Y_n\}$ is Cauchy, there exists an integer $N$ such that $\|Y_n - Y_m\| < \epsilon/3$ for all $n, m \ge N$. It follows that

$$|Y_n(t) - Y_N(t)| \le \|Y_n - Y_M\| < \epsilon/3$$

for all $n \ge N$ and all $t \in I$. Consequently

$$|Y(t) - Y_N(t)| = |\lim_{n \to \infty} Y_n(t) - Y_N(t)| = \lim_{n \to \infty} |Y_n(t) - Y_N(t)| \le \epsilon/3 ,$$

for all $t \in I$. This shows that $\|Y - Y_N\| \le \epsilon/3$. Now suppose $n > N$. Then by (3)

$$\|Y_n - Y\| \le \|Y_n - Y_N\| + \|Y_N - Y\| \le \epsilon/3 + \epsilon/3 < \epsilon .$$

This shows that $\{Y_n\}$ converges uniformly to $Y$.

We now show that $Y$ is continuous. It suffices to show that for any $t \in I$ and any $\epsilon > 0$, there is a $\delta > 0$ such that $|Y(t) - Y(s)| < \epsilon$ for all $s \in I$ with $|s - t| < \delta$.

To see this, fix $t \in I$ and choose any $\epsilon > 0$. Then choose $N$ so that $\|Y_n - Y\| < \epsilon/3$ for all $n \ge N$.

Since $Y_N$ is continuous, there is a $\delta > 0$ such $|Y_N(s) - Y_N(t)| < \epsilon/3$ for all $s \in I$ with $|t - s| < \delta$.

Now suppose that $|s - t| < \delta$ and $s \in I$. Then

$$\begin{aligned} |Y(s) - Y(t)| &= |(Y(s) - Y_N(s)) + (Y_N(s) - Y_N(t)) + (Y_N(t) - Y(t))| \\ &\le |Y(s) - Y_N(s)| + |Y_N(s) - Y_N(t)| + |Y_N(t) - Y(t)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon . \end{aligned}$$

Thus, $Y$ is continuous. □

Next consider the subset $\mathcal{F} \subset \mathcal{C}$ of continuous functions defined as follows: For fixed numbers $t_0 \in I$, $y_0$ and $b > 0$,

$$\mathcal{F} = \{Y \in \mathcal{C} \,:\, Y(t_0) = y_0, \text{ and } |Y(t) - y_0| \leq b \text{ for all } t \in I\}\,.$$

**Theorem 2.** *Let $\{Y_n\}$ be a Cauchy sequence of functions in $\mathcal{F}$. Then the limit $Y = \lim_{n\to\infty} Y_n$ is also in $\mathcal{F}$.*

**Proof.** By the previous theorem, we know that $Y$ is continuous, so we need only check that $Y(t_0) = y_0$ and $|Y(t) - y_0| \leq b$ for $t \in I$. But $Y(t_0) = \lim_{n\to\infty} Y_n(t_0) = y_0$; and since $|Y_n(t) - y_0| \leq b$ for all $t$ and all $n$, clearly

$$|Y(t) - y_0| = \lim_{n\to\infty} |Y_n(t) - y_0| \leq b\,.$$

Thus $Y \in \mathcal{F}$. □

**Definition.** A map $T : \mathcal{F} \to \mathcal{F}$ (i.e. a rule that assigns to every function in $\mathcal{F}$ another function in $\mathcal{F}$) is called a *contraction map* if there is a constant $0 < K < 1$ such that

$$\|T(Y_2) - T(Y_1)\| \leq K\|Y_2 - Y_1\|$$

for all $Y_1, Y_2 \in \mathcal{F}$.

**Theorem 3.** *Let $T : \mathcal{F} \to \mathcal{F}$ be a contraction map, let $Y_0$ be a function on $\mathcal{F}$, and let $\{Y_n\}$ be the sequence of functions in $\mathcal{F}$ defined by $Y_{n+1} = T(Y_n)$. Then $\{Y_n\}$ is a Cauchy sequence. Moreover, if $Y = \lim_{n\to\infty} Y_n$, then $T(Y) = Y$, i.e. $Y$ is a fixed point of $T$.*

**Proof.** Let $A = \|Y_1 - Y_0\|$. Since $Y_n = T^n(Y_0)$ and $Y_{n+1} = T^n(Y_1)$, we have the inequality

$$\|Y_{n+1} - Y_n| \leq K^n \|Y_1 - Y_0\| = AK^n\,.$$

Now let $m = n + k > n$ and estimate as follows:

$$\begin{aligned}
\|Y_{n+k} - Y_n\| &= \|Y_{n+k} - Y_{n+k-1} + Y_{n+k-1} - Y_{n+k-2} + \cdots + (Y_{n+1} - Y_n)\| \\
&\leq \|Y_{n+k} - Y_{n+k-1}\| + \|Y_{n+k-1} - Y_{n+k-2}\| + \cdots + \|Y_{n+1} - Y_n\| \\
&\leq AK^n(K^{k-1} + K^{k-2} + \cdots + K + 1) \leq \frac{AK^n}{1-K}
\end{aligned}$$

To see that $\{Y_n\}$ is Cauchy, choose any $\epsilon > 0$ and choose $N$ so that $\frac{AK^N}{1-K} < \epsilon$. Then by the above computation, for any $n, m \geq N$, $\|Y_n - Y_m\| < \epsilon$.

To see that $T(Y) = Y$, choose any $\epsilon > 0$. Set $Z = T(Y)$. We claim that $|Z(t) - Y(t)| < 2\epsilon$ for all $t \in I$. To see this, choose $N$ so that $\|Y - Y_N\| < \epsilon$ and $\|Y - Y_{N+1}\| < \epsilon$. It follows from this that

$$\|Z - Y_{n+1}\| = \|T(Y) - T(Y_n)\| \leq K\|Y - Y_n\| < \epsilon\,.$$

But then, $|Z(t) - Y_{n+1}(t)| < \epsilon$ and $|Y(t) - Y_{n+1}(t)| < \epsilon$. Consequently,

$$|Z(t) - Y(t)| = |Z(t) - Y_{n+1}(t) + Y_{n+1}(t) - Y(t)| \leq |Z(t) - Y_{n+1}(t)| + |Y_{n+1}(t) - Y(t)| < \epsilon + \epsilon = 2\epsilon\,.$$

Since $\epsilon$ was arbitrary, it follows that $Z(t) = Y(t)$ for all $t$. □