

Very often, it is difficult to determine the limit of a convergent sequence, yet one still wants to verify that the sequence is convergent. In this handout, we develop a criterion for convergence of a sequence that does not make direct mention of its limit.

**Definition 1.** A sequence  $\{a_n\}$  is said to be *Cauchy* (or to be a *Cauchy sequence*) if for every real number  $\epsilon > 0$ , there is an integer  $N$  (possibly depending on  $\epsilon$ ) for which

$$|a_n - a_m| < \epsilon \text{ for all } n, m \geq N. \quad (1)$$

**Theorem 1.** *A sequence is convergent if and only if it is Cauchy.*

**Proof.** ( $\Rightarrow$ ) Let  $\{a_n\}$  be a convergent sequence with limit  $L$ . To verify that  $\{a_n\}$  is Cauchy, begin by choosing a number  $\epsilon > 0$ . We must show that there is an integer  $N$  for which (1) holds.

But since  $a_n$  converges to  $L$ , there is an integer  $N > 0$  for which  $|a_n - L| < \epsilon/2$  for all  $n \geq N$ . Notice that for all  $n, m > N$  we may estimate as follows:

$$|a_n - a_m| = |(a_n - L) - (a_m - L)| \leq |a_n - L| + |a_m - L| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus  $\{a_n\}$  is Cauchy.

( $\Leftarrow$ ) Now let  $\{a_n\}$  be a Cauchy sequence. We want to show that  $\{a_n\}$  converges.

First notice that  $\{a_n\}$  is bounded. To see this, let  $\epsilon = 1$ . Then there is an integer  $N$  such that  $|a_n - a_m| < 1$ , provided  $n, m \geq N$ . Set  $m = N$  then

$$(a_N) - 1 < a_n < (a_N) + 1 \text{ for all } n \geq N.$$

Let  $U = \text{Max}\{a_1, \dots, a_N, a_N + 1\}$  and  $L = \text{Min}\{a_1, \dots, a_N, a_N - 1\}$ . Clearly,  $L \leq a_n \leq U$  for all  $n$ , so  $\{a_n\}$  is bounded.

Now let  $\{b_n\}$  and  $\{c_n\}$  be the bounded sequences defined by

$$b_n = \text{glb}_{m \geq n} a_m \quad \text{and} \quad c_n = \text{lub}_{m \geq n} a_m,$$

and notice that by construction the following inequalities are satisfied:

$$b_n \leq a_m \leq c_n \text{ for all } m \geq n. \quad (2)$$

Finally notice that  $\{b_n\}$  is nondecreasing and  $\{c_n\}$  is nonincreasing. (Exercise #29, page 532). Hence, both  $\{b_n\}$  and  $\{c_n\}$  converge (since  $\{b_n\}$  is bounded and nondecreasing and  $\{c_n\}$  is bounded and nonincreasing). Let  $B = \lim_{n \rightarrow \infty} b_n$  and  $C = \lim_{n \rightarrow \infty} c_n$ . By construction, the following inequalities hold:

$$b_n \leq B \leq C \leq c_n \text{ for all } n.$$

We will show that  $\{a_n\}$  converges by showing  $B = C$  and applying the pinching lemma.

Begin by choosing any  $\epsilon > 0$ . Then there is an integer  $N$  for which

$$|a_n - a_m| < \epsilon \text{ for all } n, m \geq N.$$

In particular,

$$a_N - \epsilon < a_m \text{ for all } m \geq N,$$

showing that  $a_N - \epsilon \leq b_N \leq B$ . Similarly

$$a_m < a_N + \epsilon \text{ for all } m \geq N,$$

showing that  $C \leq c_N \leq a_N + \epsilon$ . Hence,

$$a_N - \epsilon < B \leq C < a_N + \epsilon.$$

It follows that  $C - B < 2\epsilon$  for every  $\epsilon > 0$ , which implies that  $B = C$ .

Since  $b_n \leq a_n \leq c_n$  and  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = B$ , the Pinching Lemma applies to show that  $a_n \rightarrow B$ , concluding the proof.  $\square$