# MATH 135: COMPLEX NUMBERS

(WINTER, 2014)

The introduction of complex numbers in the 16th century made it possible to solve the equation $x^2 + 1 = 0$. These notes[1] present one way of defining complex numbers.

## 1. The Complex Plane

A complex number $z$ is given by a pair of real numbers $x$ and $y$ and is written in the form $z = x + iy$, where $i$ satisfies $i^2 = -1$. The complex numbers may be represented as points in the plane (sometimes called the Argand diagram). The real number 1 is represented by the point $(1, 0)$, and the complex number $i$ is represented by the point $(0, 1)$. The $x$-axis is called the "real axis", and the $y$-axis is called the "imaginary axis". For example, the complex numbers $3 + 4i$ and $3 - 4i$ are illustrated in Fig 1a.

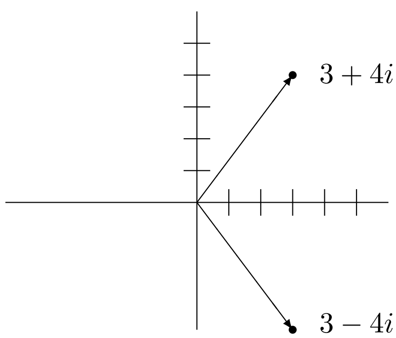


Fig 1a

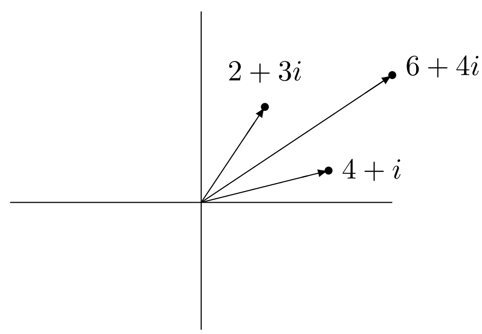


Fig 1b

Complex numbers are added in a natural way: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{1}$$

Fig 1b illustrates the addition $(4 + i) + (2 + 3i) = (6 + 4i)$. Multiplication is given by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Note that the product behaves exactly like the product of any two algebraic expressions, keeping in mind that $i^2 = -1$. Thus,

$$(2 + i)(-2 + 4i) = 2(-2) + 8i - 2i + 4i^2 = -8 + 6i$$

We call $x$ the real part of $z$ and $y$ the imaginary part, and we write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. (**Remember**: $\operatorname{Im} z$ is a *real* number.) The term "imaginary" is an historical holdover; it took mathematicians

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[1]These notes are based on notes written by Bob Phelps, with modifications by Tom Duchamp.

some time to accept the fact that $i$ (for "imaginary", naturally) was a perfectly good mathematical object. Electrical engineers (who make heavy use of complex numbers) reserve the letter $i$ to denote electric current and they use $j$ for $\sqrt{-1}$.

There is only one way we can have $z_1 = z_2$, namely, if $x_1 = x_2$ and $y_1 = y_2$. An equivalent statement (one that is important to keep in mind) is that $z = 0$ if and only if $\operatorname{Re} z = 0$ and $\operatorname{Im} z = 0$. If $a$ is a real number and $z = x + iy$ is complex, then $az = ax + iay$ (which is exactly what we would get from the multiplication rule above if $z_2$ were of the form $z_2 = a + i0$). Division is more complicated (although we will show later that the polar representation of complex numbers makes it easy). To find $z_1/z_2$ it suffices to find $1/z_2$ and then multiply by $z_1$. The rule for finding the reciprocal of $z = x + iy$ is given by:

$$\frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} \tag{2}$$

The expression $x - iy$ appears so often and is so useful that it is given a name. It is called the complex conjugate of $z = x + iy$ and a shorthand notation for it is $\overline{z}$; that is, if $z = x + iy$, then $\overline{z} = x - iy$. For example, $\overline{3 + 4i} = 3 - 4i$, as illustrated in the Fig 1a . Note that $\overline{\overline{z}} = z$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$. Exercise (3b) is to show that $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$. Another important quantity associated with a given complex number $z$ is its modulus

$$|z| = (z\overline{z})^{1/2} = \sqrt{x^2 + y^2} = \left((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\right)^{1/2}$$

Note that $|z|$ is a *real* number. For example, $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. This leads to the inequality

$$\operatorname{Re} z \leq |\operatorname{Re} z| = \sqrt{(\operatorname{Re} z)^2} \leq \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = |z| \tag{3}$$

Similarly, $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$.

**Exercises 1.**

(1) Prove that the product of $z = x + iy$ and the expression in (2) (above) equals 1.
(2) Verify each of the following:
   (a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$  (b) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = -\dfrac{2}{5}$
   (c) $\dfrac{5}{(1-i)(2-i)(3-i)} = \dfrac{1}{2}i$  (d) $(1-i)^4 = -4$
(3) Prove the following:
   (a) $z + \overline{z} = 2\operatorname{Re} z$ and $z$ is a real number if and only if $\overline{z} = z$.
   (b) $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$.
(4) Prove that $|z_1 z_2| = |z_1||z_2|$ (Hint: Use (3b).)
(5) Find all complex numbers $z = x + iy$ such that $z^2 = 1 + i$.

## 2. Polar Representation of Complex Numbers

Recall that the plane has polar coordinates as well as rectangular coordinates. The relation between the rectangular coordinates $(x, y)$ and the polar coordinates $(r, \theta)$ is

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \arctan \frac{y}{x}$$

(If $z = 0$, then $r = 0$ and $\theta$ can be anything.)

Thus, for the complex number $z = x + iy$, we can write

$$z = r(\cos\theta + i\sin\theta).$$

There is another way to rewrite this expression for $z$. Later in the course, we will show that $e^x$ can be expressed as the following power series (i.e. as an infinite sum of powers of $x$):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

For any complex number $z$, we define $e^z$ by the power series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$$

In particular,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots + \frac{(i\theta)^n}{n!} + \ldots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \end{aligned}$$

The functions $\cos\theta$ and $\sin\theta$ can also be written as power series:

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \pm \frac{(-1)^n\theta^{2n}}{(2n)!} \pm \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \pm \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} \pm \ldots$$

Thus

$$(\text{the power series for } e^{i\theta}) = (\text{the power series for } \cos\theta) + i \cdot (\text{the power series for } \sin\theta)$$

This is the Euler Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For example,

$$e^{i\pi/2} = i, \quad e^{\pi i} = -1 \quad \text{and} \quad e^{2\pi i} = +1$$

Given $z = x + iy$, then $z$ can be written in the form $z = re^{i\theta}$, where

(4) $$r = \sqrt{x^2 + y^2} = |z| \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}$$

For example the complex number $z = 8 + 6i$ may also be written as $10e^{i\theta}$, where $\theta = \arctan(.75) \approx .64$ radians. This is illustrated in Fig 2.

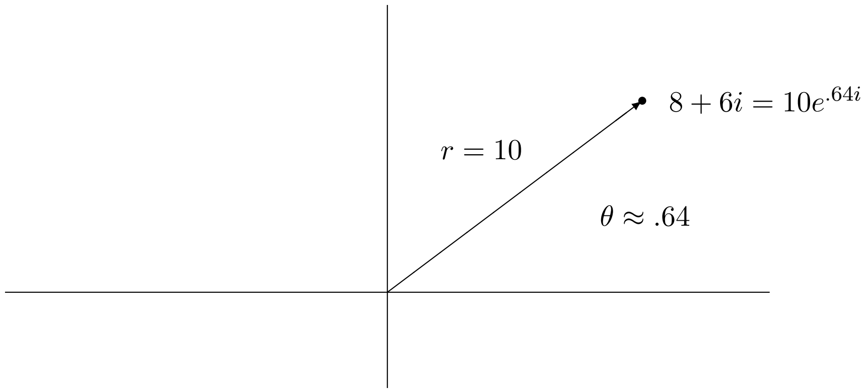


FIG 2

If $z = -4+4i$, then $r = \sqrt{4^2+4^2} = 4\sqrt{2}$ and $\theta = 3\pi/4$, therefore $z = 4\sqrt{2}e^{3\pi i/4}$. Any angle which differs from $3\pi/4$ by an integer multiple of $2\pi$ will give us the same complex number. Thus, $-4+4i$ can also be written as $4\sqrt{2}e^{11\pi i/4}$ or as $4\sqrt{2}e^{-5\pi i/4}$. In general, if $z = re^{i\theta}$, then we also have $z = re^{i(\theta+2\pi k)}$, $k = 0, \pm 1, \pm 2, \ldots$. Moreover, there is ambiguity in equation (4) about the inverse tangent which can (and *must*) be resolved by looking at the signs of $x$ and $y$, respectively, in order to determine the quadrant in which $\theta$ lies. If $x = 0$, then the formula for $\theta$ makes no sense, but $x = 0$ simply means that $z$ lies on the imaginary axis and so $\theta$ must be $\pi/2$ or $3\pi/2$ (depending on whether $y$ is positive or negative).

The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they were for rectangular coordinates. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k,\ k = 0, \pm 1, \pm 2, \ldots$. Despite this, the polar representation is very useful when it comes to multiplication:

$$\text{if } z_1 = r_1e^{i\theta_1} \quad \text{and} \quad z_2 = r_2e^{i\theta_2}, \quad \text{then} \quad z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)} \tag{5}$$

To see why this is true, write $z_1z_2 = re^{i\theta}$, so that $r = |z_1z_2| = |z_1||z_2| = r_1r_2$ (the next-to-last equality uses Ex (4a)). It remains to show that $\theta = \theta_1 + \theta_2$, that is, that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, (this is Exercise (7a)). For example, let

$$z_1 = 2+i = \sqrt{5}e^{i\theta_1}, \qquad \theta_1 \approx 0.464$$

$$z_2 = -2+4i = \sqrt{20}e^{i\theta_2}, \qquad \theta_2 \approx 2.034$$

Then $z_3 = z_1z_2$, where:

$$z_3 = -8+6i = \sqrt{100}e^{i\theta_3} \qquad \theta_3 \approx 2.498$$

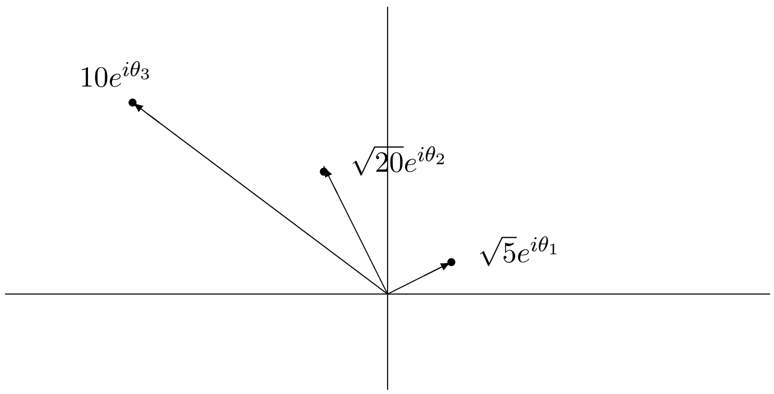


FIG 3

Applying (5) to $z_1 = z_2 = -4 + 4i = 4\sqrt{2}e^{\frac{3}{4}\pi i}$ (our earlier example), we get

$$(4 + 4i)^2 = (4\sqrt{2}e^{\frac{3}{4}\pi i})^2 = 32e^{\frac{3}{2}\pi i} = -32i.$$

By an easy induction argument, the formula in (5) can be used to prove that for any positive integer $n$

$$\text{If} \quad z = re^{i\theta}, \quad \text{then} \quad z^n = r^n e^{in\theta}$$

This makes it easy to solve equations like $z^3 = 1$. Indeed, writing the unknown number $z$ as $re^{i\theta}$, we have $r^3 e^{i3\theta} = 1 \equiv e^{0i}$, hence $r^3 = 1$ (so $r = 1$) and $3\theta = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. It follows that $\theta = 2k\pi/3$, $k = 0, \pm 1, \pm 2, \ldots$. There are only three distinct complex numbers of the form $e^{2k\pi i/3}$, namely $e^0 = 1$, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. The following figure illustrates $z = 8i = 8e^{i\pi/2}$ and its three cube roots $z_1 = 2e^{i\pi/6}$, $z_2 = 2e^{5i\pi/6}$, $z_3 = 2e^{9i\pi/6}$

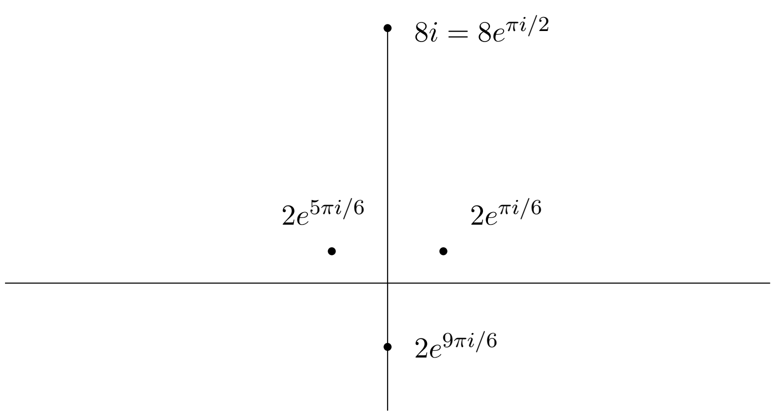


FIG 4

From the fact that $(e^{i\theta})^n = e^{in\theta}$ we obtain De Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

By expanding on the left and equating real and imaginary parts, this leads to trigonometric identities which can be used to express $\cos n\theta$ and $\sin n\theta$ as a sum of terms of the form $(\cos\theta)^j(\sin\theta)^k$. For example, taking $n = 2$ one gets $\cos 2\theta = \cos^2\theta - \sin^2\theta$. For $n = 3$ one gets $\cos 3\theta = \cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos^2\theta$.

**Exercises 2.**

(1) Let $z_1 = 3i$ and $z_2 = 2 - 2i$
  (a) Plot the points $z_1 + z_2$, $z_1 - z_2$ and $\overline{z_2}$.
  (b) Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.
  (c) Express $z_1$ and $z_2$ in polar form.

(2) Prove the following:
  (a) $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$.
  (b) Use (a) to show that $(e^{i\theta})^{-1} = e^{-i\theta}$, that is, $e^{-i\theta} e^{i\theta} = 1$.

(3) Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot $z_1$, $z_2$, $z_1 z_2$ and $z_1/z_2$.

(4) Find all complex numbers $z$ which satisfy $z^3 = -1$.

(5) Find all complex numbers $z = re^{i\theta}$ such that $z^2 = \sqrt{2} e^{i\pi/4}$.

## 3. Complex-valued Functions

Now suppose that $w = w(t)$ is a complex-valued function of the real variable $t$. That is

$$w(t) = u(t) + iv(t)$$

where $u(t)$ and $v(t)$ are real-valued functions. A complex-valued function can be thought of a defining a curve in the complex plane.

The derivative of $w(t)$ with respect to $t$ is *defined* to be the function

$$w'(t) = u'(t) + iv'(t)$$

(This is just like the definition of the derivative of a vector-valued function—just differentiate the components.) The derivative can be thought of as the tangent to the complex curve.

There are three commonly used ways to denote the derivative:

$$w'(t) = \frac{dw}{dt} = \dot{w}(t)\,.$$

We will use all three.

It is easily checked (just expand the left and right hand sides of each identity) that the following formulas hold for complex-valued functions $z = z(t)$ and $w = w(t)$:

$$\begin{aligned}
&C' = 0 \text{ where } C = \text{constant} \\
&(z + w)' = z' + w' \\
&(zw)' = z'w + zw' \\
&(Cz)' = Cz' \text{ where } C = \text{constant} \\
&(z^n)' = nz^{n-1} z'
\end{aligned}$$

One function is of particular interest to us: the *complex exponential function*. It is defined as follows:

$$e^{(\rho + i\omega)t} = e^{\rho t} e^{i\omega t} = e^{\rho t} \cos(\omega t) + ie^{\rho t} \sin(\omega t)\,.$$

Thought of as a curve in the complex plane, the complex exponential is the formula for a spiral curve. The quantity $\omega$ is the angular velocity of the spiral ($\omega > 0$ corresponds to a counterclockwise spiral, $\omega < 0$ to a clockwise one). The quantity $\rho$ measures the rate at which the spiral expands outward ($\rho > 0$) or inward ($\rho < 0$).

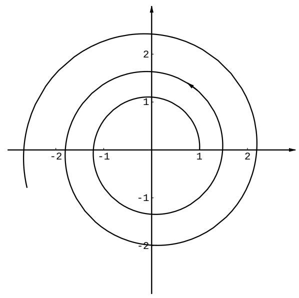

FIG 4A

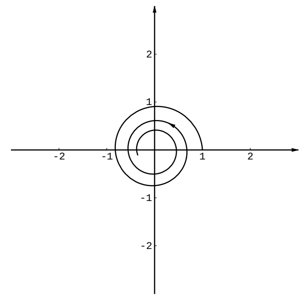

FIG 4B

Computing the derivatives of the real and imaginary parts and collecting terms yields the formula $\left(e^{(\rho+i\omega)t}\right)' = (\rho+i\omega)e^{(\rho+i\omega)t}$. In other words, *even if $r$ is a complex number, the formula*

$$\frac{d}{dt}e^{rt} = re^{rt}$$

*holds!*

## 4. THE FUNCTION $x(t) = e^{\rho t}(C_1\cos(\omega t) + C_2\sin(\omega t))$

We want to write the function

$$x(t) = C_1e^{\rho t}\cos(\omega t) + C_2e^{\rho t}\sin(\omega t)$$

in the form

$$x(t) = Ae^{\rho t}\cos(\omega t - \phi)\,,.$$

The reason for rewriting $x(t)$ in this way is that it gives an easy way to figure out what its graph looks like.

First notice that

$$Ae^{\rho t}\cos(\omega t - \phi) = (A\cos(\phi)\cos(\omega t) + A\sin(\phi)\sin(\omega t))\ e^{\rho t}$$

So

$$A\cos(\phi) = C_1 \text{ and } A\sin(\phi) = C_2\,.$$

The relation among the quantities $\rho$, $A$, $C_1$ and $C_2$ is expressed in the next figure:

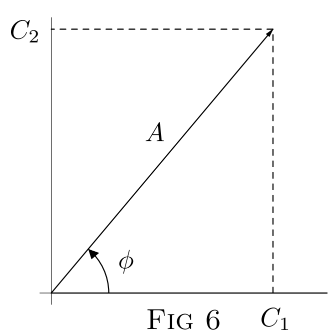


FIG 6

From Fig 6 it is clear that

$$A = \sqrt{C_1^2 + C_2^2} \text{ and } \tan(\phi) = \frac{C_2}{C_1}\,.$$

**Example 1.** Consider the function

$$x(t) = (5\cos(2t) + 4\sin(2t))e^{-t/5}\,.$$

The point $(C_1, C_2) = (5, 4)$ is in the first quadrant so $0 < \phi < \pi/2$. So

$$A = \sqrt{5^2 + 4^2} = \sqrt{41} \text{ and } \phi = \tan^{-1}(4/5)\,.$$

Hence,

$$(5\cos(2t) + 4\sin(2t))e^{-t/5} = \sqrt{41}\, e^{-t/5} \cos\left(2t - \tan^{-1}(4/5)\right)\,.$$

The sketch of this curve is:

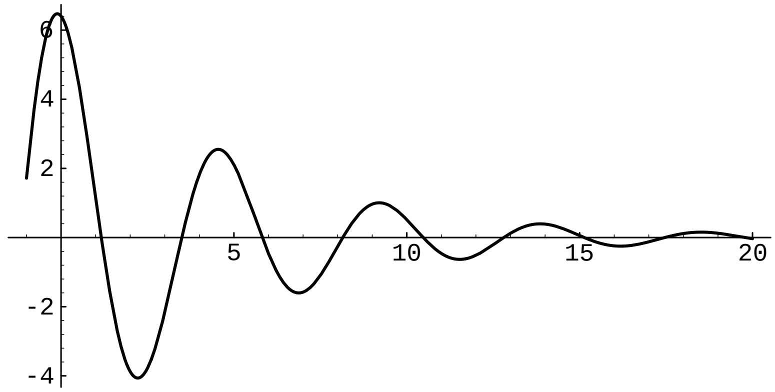


FIG 5

**Note:** There is an alternate description of $x(t)$ that makes direct used of the polar form of complex numbers. Let $C = 5 - 4i$ (note the sign change!) and $\rho + i\omega = -1/5 + 2i$. Then it is easily verified that

$$x(t) = \text{Re}\left((5 - 4i)e^{(-1/5+2i)t}\right)$$

To see that, compute as follows:

$$\begin{aligned} x(t) &= \text{Re}\left(\sqrt{41}e^{-t/5}e^{i(2t+\tan^{-1}(-4/5))}\right) \\ &= \sqrt{41}e^{-t/5}\cos(2t + \tan^{-1}(-4/5))) \\ &= \sqrt{41}e^{-t/5}\cos(2t - \tan^{-1}(4/5)) \end{aligned}$$

**Exercises 3.**

(1) Sketch the graph of the curve

$$z(t) = (2 + 2i)e^{\left(\frac{1}{2} + \pi\, i\right)t}$$

for $0 \le t \le 3$. Sketch the graph of $x = x(t) = \text{Re}(z(t))$.

(2) consider the function

$$x(t) = 3e^{-2t}\cos(4t) - 5e^{-2t}\sin(4t)\,.$$

Write it in each of the forms

$$x(t) = Ae^{\rho\, t}\cos(\omega\, t - \phi)$$

and

$$x(t) = \text{Re}\left(Ce^{rt}\right)$$

where $A$, $\omega$ and $\phi$ are real numbers and $C$ and $r$ are complex numbers.