One way that sequences are often defined is by recursion. More precisely, suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and x_0 is a real number. Then we may defined a sequence $\{x_n\}$ iteratively by the formula

$$x_{n+1} = f(x_n) \text{ for } n = 0, 1, 2, \ldots.$$

The purpose of this handout, is to give a criterion for such a sequence to converge.

We begin with some definitions. Throughout this handout, $\Omega \subset \mathbb{R}$ denotes a set of one of the forms $[a, b]$ for $a < b$, $[a, \infty)$, $(-\infty, b]$, or $\mathbb{R}$; and f denotes a function of the form

$$f : \Omega \to \Omega$$

(i.e. Ω is the domain of f and the range of f is contained in Ω.

Definition 1. The function f is said to be a *contraction map* if there is a real number K with $0 < K < 1$ for which

$$|f(x) - f(y)| \leq K|x - y| \text{ for all } x, y \in \Omega.$$

Lemma 1. *If f is a contraction map then f is continuous on Ω.*

Proof. *Exercise.*

Definition 2. A point $x_0 \in \Omega$ is called a *fixed point* of f if $f(x_0) = x_0$.

Lemma 2. *A contraction map has at most one fixed point.*

Proof. *Exercise.*

Theorem. *Let $f : \Omega \to \Omega$ be a contraction map and let $x_0 \in \Omega$. Then the sequence $\{x_n\}$ defined inductively by*

$$x_{n+1} = f(x_n)$$

is a Cauchy sequence. Moreover, the limit $x_\infty = \lim_{n\to\infty} x_n$ is a fixed point of f.

We will prove the theorem through a series of lemmas.

Lemma 3. *Suppose $x_n \to x_\infty$. Then x_∞ is a fixed point.*

Proof. First note since Ω is either a closed interval or all of $\mathbb{R}$, then $x_\infty \in \Omega$. Hence $f(x_\infty)$ is defined. (Why?)

To see that $f(x_\infty) = x_\infty$. Choose any $\epsilon > 0$. Then there is an integer $N > 0$ such that $|x_n - x_\infty| < \epsilon$ for all $n \geq N$. Choose any $n \geq N$, and notice that $x_{n+1} = f(x_n)$. Now use the triangle inequality to estimate as follows:

$$|f(x_\infty) - x_\infty| = |f(x_\infty) - f(x_n) + f(x_n) - x_\infty| \leq |f(x_\infty) - f(x_n)| + |f(x_n) - x_\infty| \leq K|x_\infty - x_n| + |x_{n+1} - x_\infty| < 2\epsilon\,.$$

Since ϵ was arbitrary, it follows that $f(x_\infty) = x_\infty$.$\square$

Lemma 4. *The sequence $\{x_n\}$ is bounded.*

Proof. Let $A = |x_1 - x_0|$. Observe that for any $n > 1$,

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2}| \leq K|x_{n-1} - x_{n-2}|\,.$$

Repeating this step n-times yields the inequality

$$|x_n - x_{n-1}| \le AK^{n-1} \text{ for all } n.$$

Thus,

$$\begin{aligned} |x_n - x_0| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + (x_1 - x_0)| \\ &\le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_1 - x_0| \\ &\le (K^{n-1} + K^{n-2} + \cdots + K + 1)A \le \frac{A}{1-K}\,. \end{aligned}$$

Let $R = A/(1-K)$, then $|x_n - x_0| \le R$ for all n. □

Lemma 5. *The sequence $\{x_n\}$ is Cauchy.*

Proof. Choose any $\epsilon > 0$. Since $0 < K < 1$, there is an integer $N > 0$ for which

$$2RK^N < \epsilon\,.$$

where R is as in the proof of the previous lemma.

We claim that $|x_n - x_m| < \epsilon$ for all $n, m \ge N$.

To see this, note first that by definition of R,

$$|x_{n-N} - x_0| \le R \text{ and } |x_{m-N} - x_0| \le R\,.$$

Hence, by the triangle inequality, $|x_{n-N} - x_{m-N}| \le 2R$. Now observe that

$$x_n = (f \circ f \circ \cdots \circ f)(x_{n-N}) \text{ and } x_m = (f \circ f \circ \cdots \circ f)(x_{m-N})\,.$$

(Here f is composed with itself N-times). Therefore,

$$|x_m - x_n| \le K^N |x_{m-N} - x_{n-N}| \le 2RK^N < \epsilon\,,$$

which is what we needed to prove.□