A Note on the Integral Test

The proof of the integral test yields more information than the test itself. The crucial point is the pair of inequalities

$$f(m+1)+\cdots+f(n) \le \int_m^n f(x)\,dx \le f(m)+\cdots+f(n-1),$$

valid whenever f is a positive, nonincreasing function on $[m,\infty)$. (Salas-Hille-Etgen does only the case $m = 1$ — it's equation (1) on p. 644 — but the argument is the same in general.) These inequalities imply that

$$\int_m^n f(x)\,dx \le f(m)+\cdots+f(n) \le f(m)+\int_m^n f(x)\,dx, \tag{1}$$

which can be used to estimate the partial sums of the series $\sum f(k)$, whether the series converges or not. If it does converge, one can let $n \to \infty$ in (1) to obtain

$$\int_m^\infty f(x)\,dx \le \sum_m^\infty f(k) \le f(m)+\int_m^\infty f(x)\,dx. \tag{2}$$

One can estimate the sum of the whole series by taking $m = 1$ in (2). For better accuracy, one can add up the first few terms by hand and use (2) to estimate the remainder.

Example. Let's estimate $\sum_1^\infty 1/k^2$. Since $\int_m^\infty x^{-2}\,dx = -x^{-1}|_m^\infty = m^{-1}$, by taking $m = 1$ in (2) we find that $1 \le \sum_1^\infty 1/k^2 \le 1+1 = 2$. We can do better by computing the sum of the first nine terms on a calculator,

$$1+\tfrac{1}{4}+\tfrac{1}{9}+\cdots+\tfrac{1}{81} = 1.5397\ldots,$$

and using (2) with $m = 10$ to get

$$0.1 = \int_{10}^\infty \frac{dx}{x^2} \le \sum_{10}^\infty \frac{1}{k^2} \le \frac{1}{10^2}+\int_{10}^\infty \frac{dx}{x^2} = 0.11,$$

with the result that

$$\sum_1^\infty \frac{1}{k^2} \approx 1.64.$$

(The exact value is $\frac{1}{6}\pi^2 \approx 1.6449341$.)