The Laplace Transform

Given $f(x)$ defined on $[0,\infty)$ $(x \geq 0)$

the Laplace Transform of $f(x)$

$$\boxed{\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dx}$$

ex ① In problem #62 Section 11.7 you computed

$$\boxed{\mathcal{L}\{x^n\} = \frac{n!}{s^{n+1}}}, \quad n \geq 0 \quad \text{when } s > 0.$$

(necessary for convergence of integral)

ex ② $\boxed{\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}, \quad s > a}$ exercise

Theorem 1: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ where $F(s) = \mathcal{L}\{f(t)\}$.

The Laplace transform is an <u>operator</u>: a function acting on functions

$A = \{f : [0,\infty) \to \mathbb{R} : \int_0^\infty e^{-sx} f(x)\,dx$ exists for $s > a\}$

$B = \{g : (a,\infty) \to \mathbb{R}$ for some $a > 0\}$

$$\begin{aligned} \mathcal{L} : A &\longrightarrow B \\ f &\longrightarrow \mathcal{L}\{f\} \\ f(x) &\longrightarrow F(s) \end{aligned}$$

It is <u>linear</u>

$$\boxed{\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}}$$

where both $\mathcal{L}\{f_1\}$ + $\mathcal{L}\{f_2\}$ exists

Follows from properties of integrals + limits

By induction you can prove $\mathcal{L}[\sum c_i f_i] = \sum c_i \mathcal{L}[f_i]$

Theorem 2 (injectivity) Let $u:[0,\infty)$ continuous. If $\mathcal{L}[u]=0$ then $u=0$.

Proof: see links.

By linearity we get that $\mathcal{L}$ is one-to-one (injective)

If $\mathcal{L}[f_1]=\mathcal{L}[f_2]$ then $\mathcal{L}[f_1-f_2]=0$ so $f_1-f_2=0$ or $f_1=f_2$.

So, we can define $\mathcal{L}^{-1}$ on the image of $\mathcal{L}$, which is <u>also</u> linear

$$\mathcal{L}^{-1}[c_1F_1(s)+c_2F_2(s)]=c_1\mathcal{L}^{-1}[F_1(s)]+c_2\mathcal{L}^{-1}[F_2(s)]$$

Before we can use the Laplace Transform on a DE we need one more computation:

Theorem 3 (derivatives) $\mathcal{L}[y^{(n)}]=s^n\mathcal{L}[y]-[y^{(n-1)}(0)+sy^{(n-2)}(0)+s^2y^{(n-3)}(0)+\dots+s^{n-1}y(0)]$.

Assume $y(x)$ is such that $\lim_{x\to\infty} e^{-sx}y^{(k)}(x)=0$, $k=0,\dots,n-1$

Proof: By induction on n

$n=0$ $\quad \mathcal{L}[y]=s^0\mathcal{L}[y]$

Assume the formula holds for $n=k$.

Then,

$$\mathcal{L}[y^{(k+1)}]=\int_0^\infty e^{-sx}y^{(n+1)}\,dx=\lim_{t\to\infty}\int_0^t e^{-sx}y^{(n+1)}\,dx$$

integration by parts

$u = e^{-sx}$ $\quad dv = y^{(k+1)}dx$

$du = -se^{-sx}$ $\quad v = y^{(k)}$

So

$$\mathcal{L}[y^{(k+1)}] = \lim_{t\to\infty}\left[e^{-sx}y^{(k)}\Big|_0^t + s\int_0^t e^{-sx}y^{(k)}\,dx\right]_\infty$$

$$= \lim_{t\to\infty}\left[e^{-st}y^{(k)}(t) - e^{0}y^{(k)}(0)\right] + s\int_0^\infty e^{-sx}y^{(k)}\,dx$$ evaluating the first part

$$= s\int_0^\infty e^{-sx}y^{(k)}\,dx - y^{(k)}(0)$$

$$= s\,\mathcal{L}[y^{(k)}] - y^{(k)}(0)$$

$$= s\left[s^k\mathcal{L}[y] - \sum_{k=1}^{n} s^{k-1}y^{(n-k)}(0)\right] - y^{(k)}(0)$$ by induction hyp.

$$= s^{k+1}\mathcal{L}[y^{(k)}] - \sum_{k=0}^{n} s^k y^{(n-k)}(0)$$ which is the formula with $n=k+1$

example ③ Solve $y'' - 5y' + 6y = 3e^{7t}$, $y(0)=1$, $y'(0)=2$

using the Laplace Transform.

$$\mathcal{L}[y'' - 5y' + 6y] = \mathcal{L}[3e^{7t}]$$

$$s^2\mathcal{L}[y] - y'(0) - sy(0) - 5\left(s\mathcal{L}[y] - y(0)\right) + 6\mathcal{L}[y] = 3\cdot\frac{1}{s-7}$$

$$\mathcal{L}[y](s^2 - 5s + 6) = \frac{3}{s-7} + y'(0) + sy(0) + 5y(0)$$

$$= \frac{3}{s-7} + 2 + s + 5 = \frac{3}{s-7} + 7 + s$$

$$\mathcal{L}[y] = \frac{3}{(s-7)(s^2-5s+6)} + \frac{7}{s^2-5s+6} + \frac{s}{s^2-5s+6}$$

PARTIAL FRACTIONS

$$= \frac{3}{(s-7)(s-2)(s-3)} + \frac{7}{(s-2)(s-3)} + \frac{s}{(s-2)(s-3)}$$

remember the cover up method.
you can use complex numbers for values of s

$$= \frac{\frac{3}{5\cdot 4}}{s-7} + \frac{\frac{3}{-5\cdot(-1)}}{s-2} + \frac{\frac{3}{-4\cdot 1}}{s-3} + \frac{\frac{7}{-1}}{s-2} + \frac{\frac{7}{1}}{s-3} + \frac{\frac{2}{2-3}}{s-2} + \frac{\frac{3}{3-2}}{s-3}$$

$$= \frac{3/20}{s-7} + \frac{\frac{3}{5}-7-2}{s-2} + \frac{-\frac{3}{4}+7+3}{s-3}$$

$$= \frac{3/20}{s-7} + \frac{-42/5}{s-2} + \frac{37/4}{s-3}$$

take inverse

$$y = \mathcal{L}^{-1}\left[\frac{3/20}{s-7} + \frac{-42/5}{s-2} + \frac{37/4}{s-3}\right]$$

$$= \frac{3}{20}\mathcal{L}^{-1}\left[\frac{1}{s-7}\right] - \frac{42}{5}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{37}{4}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

$$= \frac{3}{20}e^{7t} - \frac{42}{5}e^{2t} + \frac{37}{4}e^{3t}$$

Which types of DE's can we work with?

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = f(t)$$

For the method of undetermined coefficients,
$f(t) = P(t)e^{at}$, $P(t)\sin at$, $P(t)\cos at$, $P(t)e^{at}\sin bt$, $P(t)e^{at}\cos bt$ where $P(t)$ is a polynomial (usually just 1 or t)

For the variation of parameters method,
$n=2$, a_2, a_1, a_0 can be replaced by functions $a_i(t)$
and $f(t)$ could be cts function (in theory, messy
$f(t)$'s, $a_2(t), a_1(t), a_0(t)$'s will make the integrals
difficult)
continuity is required because we end up
integrating their combinations.

For the Laplace Transform, $a_i \in \mathbb{R}$
and $f(t)$ must be a function whose $\mathcal{L}[f]$ exists.
One guarantee would be:
f is cts and $|f(t)| \leq Ae^{-bt}$ for some $b>0$, $t \geq t_0$
f is of exponential order
(f is of exponential order)
All functions listed for UC method are.

ex 4 $f(t)=e^{t^2}$ is not. (exercise).

We can relax the continuity requirement on f
because we can also integrate piecewise
continuous functions:

ex 5 $f(t)=\begin{cases} t, & t \geq 5 \\ 3 & 0 \leq t < 5 \end{cases}$

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^5 e^{-st} \cdot 3\,dt + \int_5^\infty e^{-st} \cdot t\,dt$$

$$= \int_0^5 e^{-st} \cdot 3\,dt + \int_0^\infty e^{-st} \cdot t\,dt - \int_0^5 e^{-st} \cdot t\,dt$$

$$= \int_0^5 e^{-st}(3-t)\,dt + \mathcal{L}[t]$$

$3-t$	e^{-st}
-1	$\frac{e^{-st}}{-s}$
0	$\frac{e^{-st}}{s^2}$

skip

easier way.

$$= e^{-st}\left[\frac{3-t}{-s} + \frac{1}{s^2}\right]\Bigg|_0^5 + \mathcal{L}[t]$$

$$= e^{-5s}\left[\frac{-2}{-s} + \frac{1}{s^2}\right] - e^0\left[\frac{3}{-s} + \frac{1}{s^2}\right] + \mathcal{L}[t]$$

$$= e^{-5s}\left(\frac{1}{s^2} + \frac{2}{s}\right) - \frac{1}{s^2} + \frac{3}{s} + \frac{1}{s^2}$$

$$= e^{-5s}\left[\frac{1}{s^2} + \frac{2}{s}\right] + \frac{3}{s}$$

Or, in its most general form, any f where $\int_0^\infty e^{-st} f(t)\,dt$ makes sense should work.

example 6 $f(t) = t^{-1/2}$, which is not ets on $[0,\infty)$ or any $[\varepsilon,\infty)$. but ets on $(0,\infty)$

$$\mathcal{L}[t^{-1/2}] = \int_0^\infty e^{-st}\, t^{-1/2}\,dt$$

$st = y^2$, $\quad \frac{s}{y^2} = \frac{1}{t}$

$dt = \frac{1}{s} 2y\,dy$

$$= \lim_{a\to\infty} \int_0^{\sqrt{s}\sqrt{a}} e^{-y^2} \frac{s^{1/2}}{y} \frac{1}{s} 2y\,dy \quad , s>0$$

$$= 2s^{-1/2} \lim_{a\to\infty} \int_0^{\sqrt{s}\sqrt{a}} e^{-y^2}\,dy$$

$$= 2s^{-1/2} \underbrace{\int_0^\infty e^{-y^2}\,dy}_{\text{later when we do polar coordinates!}} = 2s^{-1/2}\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{s}}$$

To make this method quick, we need to be able to do $\mathcal{L}$ + $\mathcal{L}^{-1}$ fast $\rightarrow$ table

Before doing $\mathcal{L}^{-1}$, there will be algebra involved (lots of partial fractions!)

Ideally, we don't want to do $\int_0^\infty e^{-st} f(t)\,dt$ integral every time (ever?)

Piecewise Defined Functions

back to $f(t) = \begin{cases} t, & t \geq 5 \\ 3 & 0 \leq t < 5 \end{cases}$

or generalize: $f(t) = \begin{cases} f_1(t) & t \geq c \\ f_2(t) & 0 \leq t < c \end{cases}$

$\alpha : \mathbb{R} \rightarrow \mathbb{R}$

$t \rightarrow 0 \quad t < 0$

$t \rightarrow 1 \quad t \geq 1$

shift by c $\quad \alpha_c : \mathbb{R} \rightarrow \mathbb{R}$

$t \rightarrow 0 \quad t < c$

$t \rightarrow 1 \quad t \geq c$

$\boxed{\alpha_c(t) = \alpha(t-c)}$

Given $g : [0,\infty) \rightarrow \mathbb{R}$ we can shift g

$g(t)$

$g(t-c)\,\alpha_c(t) = g(t-c)\,\alpha(t-c)$

Compute the Laplace Transform of the shift

$$\int_0^\infty e^{-st} g(t-c)\alpha(t-c)\,dt \qquad \begin{cases} t-c=u \\ t=u+c \\ dt=du \end{cases}$$

$\alpha_c = 0$ when $t<c$

$$= \int_c^\infty e^{-st} g(t-c)\,dt$$

$$= \int_0^\infty e^{-s(u+c)} g(u)\,du = e^{-sc}\int_0^\infty e^{-su} g(u)\,du$$

$$= e^{-sc}\,\mathcal{L}[g]$$

Thm 4 Shifts

$$\boxed{\mathcal{L}[g(t-c)\alpha(t-c)] = e^{-cs}\,\mathcal{L}[g]}$$

How does this work with piecewise functions?

ex: $f(t) = \begin{cases} 3 & 0 \le t \le 5 \\ t & t \geq 5 \end{cases}$

$1-\alpha(t-c) = \begin{cases} 0 & t \geq c \\ 1 & 0<t<c \end{cases}$

knocks out $t \geq c$

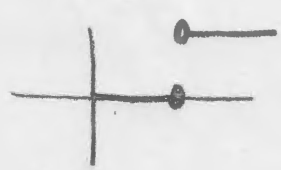
$f(t) = 3(1-\alpha(t-5)) + t(\alpha(t-5))$

$\mathcal{L}[3(1-\alpha(t-5))] = 3(\mathcal{L}[1] - \mathcal{L}[\alpha(t-5)])$

$g=1$

$= 3\cdot\frac{1}{s} - 3e^{-5s}\,\frac{1}{s}$

$\alpha(t-c)$



knocks out $t \le c$

However, $t\alpha(t-5)$ does not look like $g(t-5)\alpha(t-5)$

$t\alpha(t-5) = [(t+5)-5]\alpha(t-5)$ now it does.

$$\mathcal{L}[((t+5)-5)u(t-5)] = e^{-5s}\mathcal{L}[t+5]$$

$g(t)=t+5$

$$= e^{-5s}\left[\frac{1}{s^2}+\frac{5}{s}\right]$$

So $$\mathcal{L}[f(t)] = e^{-5s}\left(\frac{1}{s^2}+\frac{5}{s}\right)+3\left(\frac{1}{s}-\frac{e^{-5s}}{s}\right)$$

$$= e^{-5s}\left(\frac{1}{s^2}+\frac{5}{s}-\frac{3}{s}\right)+\frac{3}{s}$$

$$= e^{-5s}\left(\frac{1}{s^2}+\frac{2}{s}\right)+\frac{3}{s}.$$

example 7 messier 3-part function:

$$f(t) = \begin{cases} e^{4t} & 0 \le t < 7 \\ 6 & 7 \le t < 12 \\ t^2 & t \geq 12 \end{cases}$$

$$f(t) = e^{4t}\left(1-u(t-7)\right) + \underset{\substack{\uparrow \\ \text{knockout} \\ t \le 7}}{6u(t-7)}\,\underbrace{\left(1-u(t-12)\right)}_{\substack{\uparrow \\ \text{knock} \\ \text{out } t \geq 12}} + t^2 u(t-12)$$

$$= e^{4t} + e^{4(t+7-7)}u(t-7) + 6u(t-7) - 6u(t-7)u(t-12) + (t-12+12)^2 u(t-12)$$

$$= \underset{g(t)=e^{4t}}{e^{4t} + e^{28}e^{4(t-7)}u(t-7)} + \underset{g(t)=6}{6u(t-7)} - \underset{g=-6}{6u(t-12)}$$

$$+ \left[(t-12)^2+24(t-12)+144\right]u(t-12)$$

$g(t)=t^2+24t+144$

$$\mathcal{L}[f(t)] = \frac{1}{s-4} + \frac{e^{28}e^{-7s}}{s-4} + \frac{6}{s}e^{-7s} - \frac{6e^{-12s}}{s} + e^{-12s}\left[\frac{2}{s^3}+\frac{24}{s^2}+\frac{144}{s}\right]$$

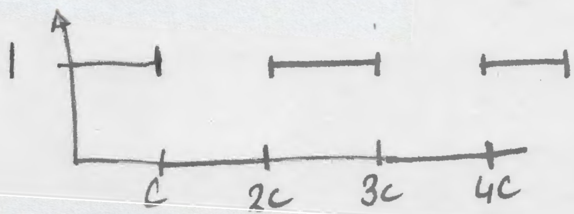
Laplace Transform of Periodic Functions

The Laplace Transform also works well with periodic functions

fixed w, $\boxed{f(t+w)=f(t)}$ for all t

ex 8 $f(t)=\sin t$ with $w=2\pi$

ex 9 pulse function



$w=2c$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{\int_0^w e^{-st} f(t)\,dt}{1-e^{-sw}}}$$

HW?

For ex 9 $\displaystyle\int_0^{2c} e^{-st} f(t)\,dt = \int_0^c e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_0^c = \frac{e^{-cs}}{-s} - \frac{1}{-s}$

$$= \frac{1}{s}\left(1-e^{-cs}\right)$$

so $\displaystyle\mathcal{L}\{f\} = \frac{1-e^{-cs}}{(1-e^{-sw})s}$

HW ex 10 $y''+y=f(t)$ $\quad y(0)=1 \quad y'(0)=0 \quad f(t)=\begin{cases} 0 & 0\le t < t_0 \\ \varepsilon & t_0 \le t \le t_1 \\ 0 & t_1 \le t \le 2\pi \end{cases}$

& repeat

$$\mathcal{L}\{f\} = \frac{\int_0^{2\pi} e^{-st} f(t)\,dt}{1-e^{-2\pi s}} = \frac{\int_0^{\varepsilon} e^{-st}\,dt}{1-e^{-2\pi s}} = \frac{1-e^{-\varepsilon s}}{(1-e^{-2\pi s})s}$$

$$s^2\mathcal{L}\{y\} - \left(y'(0) - s\,y(0)\right) + \mathcal{L}\{y\} = \frac{1-e^{-\varepsilon s}}{s(1-e^{-2\pi s})}$$

$$(s^2+1)\mathcal{L}\{y\} = \frac{1-e^{-\varepsilon s}}{s(1-e^{-2\pi s})} - s$$

$$\mathcal{L}\{y\} = \frac{1-e^{-\varepsilon s}}{s(s^2+1)(1-e^{-2\pi s})} - \frac{s}{s^2+1}$$

$f_1(s) = \mathcal{L}\{f(t)\}$

$f_2(s) = \frac{1}{s^2+1} = \mathcal{L}\{\sin t\}$

Two more theorems before we can practice $\mathcal{L}$ and $\mathcal{L}^{-1}$.

Thm 5 derivatives of $F(s)$

If $\mathcal{L}\{f(t)\} = F(s)$

then $F'(s) = \mathcal{L}\{-t f(t)\}$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{a\to\infty} \int_0^a e^{-st} f(t)\,dt = F_a(s)$$

$$F_a'(s) = \lim_{a\to\infty} \int_0^a (-t) e^{-st} f(t)\,dt$$

More generally,

$$\frac{d^{(n)}}{ds^n} F(s) = \mathcal{L}\{(-t)^n F(t)\}$$

ex 11

$$\mathcal{L}\{t^3 e^t\} = -\mathcal{L}\{(-t)^3 e^t\}$$

$$= \frac{d^3}{ds^3}\mathcal{L}\{e^t\} = \frac{d}{ds^3}\frac{1}{s-1}$$

$$= \frac{(-1)(-2)(-3)}{(s-1)^4} = \frac{-6}{(s-1)^4}$$

Theorem 5 $\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t g(\beta) f(t-\beta)\,d\beta$

where $\mathcal{L}\{f\}=F$, $\mathcal{L}\{g\}=G$.

Proof: Let $F(s)=\int_0^\infty e^{-st} f(t)\,dt = \lim_{a\to\infty}\int_0^a$

$$G(s)=\int_0^\infty e^{-s\beta} g(\beta)\,d\beta = \lim_{b\to\infty}\int_0^b$$

then,

$$F(s)G(s) = \lim_{b\to\infty} F(s)\int_0^b e^{-s\beta} g(\beta)\,d\beta$$

$$= \lim_{b\to\infty}\int_0^b \underline{F(s)e^{-s\beta}}\, g(\beta)\,d\beta$$

$$\mathcal{L}^{-1}(F(s)e^{-s\beta}) = f(t-\beta)\,d(t-\beta)$$

$$F(s)e^{-s\beta} = \int_0^\infty e^{-st} f(t-\beta)\,d(t-\beta)\,dt$$

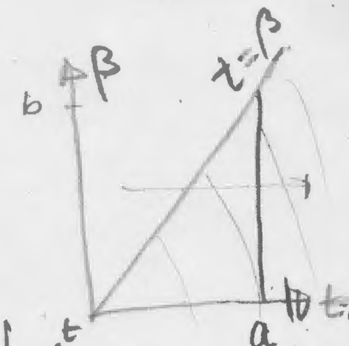
$$= \lim_{b\to\infty}\lim_{a\to\infty}\int_0^b\int_0^a e^{-st} f(t-\beta)\,d(t-\beta)\,dt \cdot g(\beta)\,d\beta$$

$$= \lim_{b\to\infty}\lim_{a\to\infty}\int_0^b\int_\beta^a e^{-st} f(t-\beta)g(\beta)\,dt\,d\beta$$

since $d(t-\beta)=0$ when $t<\beta$

Double Integrals skip for now

$$= \lim_{a,b}\int_0^a\int_0^t e^{-st} f(t-\beta)g(\beta)\,d\beta\,dt$$



$$= \lim_{a}\int_0^a e^{-st}\cdot\int_0^t g(\beta)f(t-\beta)\,d\beta\;dt = \mathcal{L}\left\{\int_0^t g(\beta)f(t-\beta)\,d\beta\right\}.$$

$Res=1\ y(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,y(1-s)$

Bonus: The Γ-function

$$\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,dx\ ,\quad \Gamma(1)=1$$

HW? $\Gamma(s+1)=\lim_{a\to\infty}\int_0^a x^{(s+1)-1}e^{-x}\,dx$

$$=\lim_{a\to\infty}\left.-x^se^{-x}\right|_0^a+s\int_0^a x^{s-1}e^{-x}dx \qquad \begin{array}{ll} u=x^s & dv=e^{-x} \\ du=sx^{s-1} & v=-e^{-x}\end{array}$$

$$=\lim_{a\to\infty}\left(-a^se^{-a}\right)+s\Gamma(s)$$

for any s

$$=s\Gamma(s)$$

From $\Gamma(1)=1$ we get $\Gamma(s+1)=s!$ if $s=0,1,2,3,\ldots$

Fact: $\Gamma\left(\frac{1}{2}\right)=\int_0^\infty x^{-1/2}e^{-x}dx=\sqrt{\pi}$

$-2\Gamma\left(-\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)$ so $\Gamma\left(-\frac{1}{2}\right)=-\frac{\sqrt{\pi}}{2}$

ex 12 $\mathcal{L}\{t^{-1/2}e^{at}\}=\lim_{b\to\infty}\int_0^b e^{-(s-a)t}t^{-1/2}dt \qquad x=(s-a)t$

$$=\lim_{b\to\infty}\int_0^b e^{-x}\left(\frac{x}{s-a}\right)^{-1/2}dx$$

$$=\frac{1}{\sqrt{s-a}}\int_0^\infty e^{-x}x^{-1/2}dx=\frac{\sqrt{\pi}}{\sqrt{s-a}}$$