Solutions to Math 308 G Spring 2016 Midterm 1

1. Solve the following system of three equations with five unknowns.

$$\begin{aligned} 2x_1 &+ x_2 - 3x_5 = -2 \\ x_2 &+ x_4 + 5x_5 = 6 \\ -3x_1 &+ x_3 + 2x_5 = -1 \end{aligned}$$

If the system is consistent, identify your particular solution and the homogenous solution.

$$\left[\begin{array}{cccccc} 2 & 1 & 0 & 0 & -3 & -2 \\ 0 & 1 & 0 & 1 & 5 & 6 \\ -3 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \times 2 \rightarrow \left[\begin{array}{cccccc} 2 & 1 & 0 & 0 & -3 & -2 \\ 0 & 1 & 0 & 1 & 5 & 6 \\ -6 & 0 & 2 & 0 & 4 & -2 \end{array}\right] \times 3$$

$$\rightarrow \left[\begin{array}{cccccc} 2 & 1 & 0 & 0 & -3 & -2 \\ 0 & 1 & 0 & 1 & 5 & 6 \\ 0 & 3 & 2 & 0 & -5 & -8 \end{array}\right] \begin{array}{c} -1 \\ -3 \end{array} \rightarrow \left[\begin{array}{cccccc} 2 & 0 & 0 & -1 & -8 & -8 \\ 0 & 1 & 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & -3 & -20 & -26 \end{array}\right] \begin{array}{c} \times\frac{1}{2} \\ \\ \times\frac{1}{2} \end{array}$$

$$\rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -1/2 & -4 & -4 \\ 0 & 1 & 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & -3/2 & -10 & -13 \end{array}\right]$$

$x_4 + x_5$ are free

Let $x_4 = s_1$ $x_5 = s_2$

Row 3:
$x_3 - \frac{3}{2}x_4 - 10x_5 = -13 \longrightarrow x_3 = \frac{3}{2}s_1 + 10s_2 - 13$

Row 2:
$x_2 + x_4 + 5x_5 = 6 \longrightarrow x_2 = -s_1 - 5s_2 + 6$

Row 1:
$x_1 - \frac{1}{2}x_4 - 4x_5 = -4 \longrightarrow x_1 = \frac{1}{2}s_1 + 4s_2 - 4$.

The solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s_1 + 4s_2 - 4 \\ -s_1 - 5s_2 + 6 \\ \frac{3}{2}s_1 + 10s_2 - 13 \\ s_1 \\ s_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 \\ 6 \\ -13 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ -1 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} s_1 + \begin{bmatrix} 4 \\ -5 \\ 10 \\ 0 \\ 1 \end{bmatrix} s_2}_{\vec{x}_h}$$

2. Determine if the following set of vectors are independent. If they are not, write one of them as a linear combination of the other.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

Solve $A\vec{x}=0$, columns of A are the given vectors

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ -1 & 3 & -2 & 3 & 0 \\ 4 & -1 & 2 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \times 1 \\ \times -4 \\ \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 7 & -6 & 8 & 0 \end{array}\right] \begin{array}{l} \\ -1 \\ -7 \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 15 & 15 & 0 \end{array}\right] -5 \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \times \frac{1}{3}$$

echelon form

x_4 is free so there are infinitely many solutions to $A\vec{x}=\vec{0}$. The vectors are not linearly independent.

Carrying on:

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2 \\ 3 \\ \\ \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] 2$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

reduced echelon form

$x_1 + x_4 = 0 \rightarrow x_1 = -s$

$x_2 + 2x_4 = 0 \rightarrow x_2 = -2s$

$x_3 + x_4 = 0 \rightarrow x_3 = -s$

$x_4 = s$

$$\vec{x} = \begin{bmatrix} -s \\ -2s \\ -s \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

Take $s=1$, for example

$$-1\begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 3 \\ 4 \end{bmatrix} = 0 \text{ so}$$

$$\begin{bmatrix} -1 \\ -1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \\ 2 \end{bmatrix}$$

3. Answer the questions about the following linear transformation.

$$T: \quad \mathbb{R}^2 \quad \to \quad \mathbb{R}^3$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \to \quad \begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 4x_2 \\ x_1 - 2x_2 \end{bmatrix}$$

(a) What are the domain and codomain of T? Which matrix represents T?

Domain: $\mathbb{R}^2$ Matrix $\begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 1 & -2 \end{bmatrix}$

Codomain: $\mathbb{R}^3$

(b) Is T onto?

No, T cannot be onto by a Thm. from Chapter 3:
Any $T: \mathbb{R}^m \to \mathbb{R}^n$ cannot be onto when $m < n$.
In this case, $m = 2 < n = 3$.

(c) Is T one-to one?

Yes. Solve $T\vec{x} = 0$

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ -1 & 4 & 0 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{matrix} \\ \times\frac{1}{2} \\ \times\frac{1}{5} \end{matrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 = x_2 = 0 \quad \text{so } T \text{ is one-to-one}$$

<u>OR</u> the vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ are linearly independent since one is not a scalar multiple of the other.

So $x_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = \vec{0}$ has the unique solution $x_1 = x_2 = 0$

and T is one-to-one.

4. (a) True or false? A set of two vectors is linearly dependent if and only if one is a scalar multiple of the other. Explain.

True. Linearly dependent means $c_1\vec{u_1}+c_2\vec{u_2}=\vec{0}$ has a non-zero solution c_1, c_2. Then we can write $\vec{u_1}=-\frac{c_2}{c_1}\vec{u_1}$ or $\vec{u_2}=-\frac{c_1}{c_2}\vec{u_1}$ since one of c_1 or c_2 is not zero.
One is a scalar multiple of the other means $\vec{u_1}=\alpha\vec{u_2}$ or $\vec{u_2}=\beta\vec{u_1}$. Then we have $\vec{u_1}-\alpha\vec{u_2}=\vec{0}$ or $-\beta\vec{u_1}+\vec{u_2}=\vec{0}$. so they are dependent.

(b) True or False? If $m > n$, any set of m vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$. Explain.

False. Let $m=3$, $n=2$
Take vectors $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}$ which do not span all of $\mathbb{R}^2$ (they span the line through $(0,0)$ and $(1,1)$)

(c) Give an example of a non zero linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is not one-to-one. If such an example does not exist, explain why not.

$T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $A = \begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}$

$T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right)=\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}2 & -2\\6 & -6\end{bmatrix}=\vec{0}$ so T is not one-to-one.

(d) Give an example of a set of three vectors which are linearly dependent and span $\mathbb{R}^3$. If such an example does not exist, explain why not.

Not possible. By the Big Theorem, n vectors span $\mathbb{R}^n$ if and only if they are independent. In this case $n=3$.

(e) Give an example of a linear system with fewer variables than equations that has infinitely many solutions. If such an example does not exist, explain why not.

$x_1+x_2=1$
$2x_1+2x_2=2$
$3x_1+3x_2=3$

$\vec{x}=\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}1\\-1\end{bmatrix}s$