Final Review Problems

1. As part of my solution, I reduced the following matrix:

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 4 & -8 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 2 & -7 & 5 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   What was the question? What was the answer?

   Try to come up with as many different questions as you can think of. Reducing a matrix is the most commonly used tool in linear algebra.

2. Solve the following systems identifying the particular solution and the homogeneous solution: Problems 19, 23, 25 in Section 1.2.

3. The set of solutions to a homogeneous system is a subspace. For the following systems, find a basis for the subspace of solutions and its orthogonal complement. Sketch the solution space and its orthogonal complement showing the basis vectors. The first is in $\mathbf{R}^2$, the last two are in $\mathbf{R}^3$ so they will require more artistic skills.

   (a) $2x_1 + 3x_2 = 0$.

   (b) $3x_1 - 5x_2 - 2x_3 = 0$

   (c) $4x_1 + 2x_2 - x_3 = 0$ and $x_1 - x_2 + 3x_3 = 0$.

4. Consider the linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^4$ given by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & -2 & 0 & 2 & -2 \end{bmatrix}.$$

   Find bases for $\text{Ran}(T) = \text{Col}(A)$, $\text{Ker}(T) = \text{Null}(A)$ and their orthogonal components. Give the dimensions of these four subspaces and say whether they are subspaces of $\mathbb{R}^5$ or $\mathbb{R}^4$.

5. Compute the determinant of the matrix

$$\begin{bmatrix} 17 & 10 & 3 \\ 12 & 0 & 8 \\ 18 & -36 & -27 \end{bmatrix}$$

   by first using row reduction to simplify the determinant computation.

6. Compute the inverses of the following matrices: Questions 5, 11, 15 in Section 3.3. For the two by two matrices, it is faster the use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

7. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be a map given by $T(\mathbf{x}) = \mathbf{proj_v x}$.

   (a) Let $n = 3$ and $\mathbf{v} = [1\,3\,5]^T$. Show that T is a linear transformation by showing it satisfies the two properties in the definition of a linear transformation. Then, find the 3×3 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

   (b) Show that T (with arbitrary n and $\mathbf{v}$) is a linear transformation by showing it satisfies the two properties in the definition of a linear transformation. What is the range of T? What is the kernel of T? Look above for an example and think geometrically.

8. Let A be an $n \times n$ matrix such that $A = -A^T$. Show that if n is odd then $\det A = 0$. Show by example that $\det A$ does not have to be 0 if n is even.

9. Let A be a 3×3 matrix with eigenvalues $\lambda_1 = 3$, $\lambda_2 = -3$, and $\lambda_3 = 0$ and that

$$\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix},$$

are the corresponding (unit length) eigenvectors. Find A.

10. Question 5 in Section 3.2. Also, If T is a transformation given by the matrix C and S is a transformation given by the matrix E, what are the domain and codomain of the transformations T, S and $S \circ T$? Is S one-to-one or onto? What is the matrix for the transformation $S \circ T$. Can we form the composition $T \circ S$? Why or why not?

11. Questions 5, 9, 19 in Section 6.3.

12. Question 13 in Section 6.4 by giving the change of basis matrix P, its inverse P^{-1}, the diagonal matrix D and verifying the product $A = PDP^{-1}$.

13. Compute A^{10} for

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$$

by first diagonalizing A.

14. Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find a matrix B such that $B^2 = A$ by diagonalizing A first.

15. Questions 9 and 11 in Section 8.5.