Solutions to Theory Review Questions

1(a) This is only possible when $n=m$.

When $n=m=1$, take $\vec{u}_1=[1]$ or any $[a]$ with $a\neq 0$.

When $n=m=2$, take $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ or any two non-zero vectors which are not parallel.

When $n=m=3$, take $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

In general, any $\vec{u}_1,\vec{u}_2,\vec{u}_3$ such that the reduced echelon form of the matrix $[\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$ is $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ will work. So you can start with $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, do a bunch of row operations and use the columns of the result for $\vec{u}_1,\vec{u}_2,\vec{u}_3$.

The case $n=m=4$ is similar to the case $n=m=3$.

1(b) If the set spans $\mathbb{R}^n$, you must have $m\geq n$.

When $n=m$, you can't have the vectors dependent (see the BIG THEOREM).

So in this case $m>n$.

$(m,n)=(2,1)$ take $\{[a],[b]\}$ with $a\neq 0$, $b\neq a$

e.g. $\{[1],[2]\}$ or $\{[1],[0]\}$.

for $(m,n)=(3,1)$ and $(4,1)$, throw some more vectors into the above set.

$(m,n)=(3,2)$, $\{\vec{u}_1,\vec{u}_2,\vec{u}_1+\vec{u}_2\}$ with $\vec{u}_1$ + $\vec{u}_2$ nonparallel + nonzero works.

$(m,n)=(4,2)$ $\{\vec{u}_1,\vec{u}_2,\vec{u}_1+\vec{u}_2$, some other arbitrary vector$\}$ like above.

$(m,n)=(4,3)$ Take any 3 independent vectors which span $\mathbb{R}^3$, for example $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ and throw in <u>any</u> fourth vector.

1(c) In this case we must have $m \leq n$ so they are linearly independent. To make sure they don't span $\mathbb{R}^n$, we must have $m \neq n$. So in this case $m < n$.

$(m,n) = (1,2)$ or $(1,3)$ or $(1,4)$
just take one non-zero vector in $\mathbb{R}^2, \mathbb{R}^3$ or $\mathbb{R}^4$.

$(m,n) = (2,3)$ or $(2,4)$
take two non-zero non-parallel vectors
(with two vectors, being independent is one not being a scalar multiple of the other)

$(m,n) = (3,4)$ any 3 linearly independent vectors

For example, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

1(d) when $m=1$, take $\{\vec{0}\}$ in all of $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$
when $m \geq 2$, the case with $n=1$ won't work since the moment we include $[a] \neq \vec{0}$ in the set of vectors, $\{\vec{0}, [a]\}$ will span $\mathbb{R}$.

For $n \geq 2$, when $m=2$ take $\{\vec{0}, \vec{u}\}$ with $\vec{u} \neq \vec{0}$ in $\mathbb{R}^n$.
when $m=3$, take $\{\vec{0}, \vec{u}, 2\vec{u}\}$ with $\vec{u} \neq \vec{0}$ in $\mathbb{R}^n$.
when $m=4$, take $\{\vec{0}, \vec{u}, 2\vec{u}, 3\vec{u}\}$ with $\vec{u} \neq \vec{0}$ in $\mathbb{R}^n$.

In $m=1$, $\{\vec{0}\}$ is the only possibility. Why?
For $n \geq 2, m \geq 2$; you can come up with other examples.

2(a) This will only work when $n=m$.
In all, you can use $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (the identity map)
$$\vec{x} \longrightarrow \vec{x}$$

<u>OR</u> you can take your answers from 1(a)
say $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ in $\mathbb{R}^3$
and form $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$$\begin{bmatrix}1\\0\\0\end{bmatrix} \longrightarrow \vec{u_1}$$
$$\begin{bmatrix}0\\1\\0\end{bmatrix} \longrightarrow \vec{u_2}$$
$$\begin{bmatrix}0\\0\\1\end{bmatrix} \longrightarrow \vec{u_3}$$

i.e. make your answers from 1(a) the columns of the matrix for T. (This is how the two questions are related.)

2 (b) Here we must have $m > n$.
Again, take your answers from 1(b) and make them the columns for the matrix of T.

2 (c) We must have $m < n$, take your answers from 1(c) + make them the columns of the matrix for T.

2 (d) Take $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$, the zero map.
$$\vec{x} \longrightarrow \vec{0}$$

There are other possible answers but this works for all (m,n).