What is a subspace and what is not?

The definition of a subspace is a subset S of some $\mathbb{R}^n$ such that whenever $\mathbf{u}$ and $\mathbf{v}$ are vectors in S, so is $\alpha\mathbf{u} + \beta\mathbf{v}$ for any two scalars (numbers) α and β. However, to identify and picture (geometrically) subspaces we use the following theorem:

Theorem: A subset S of $\mathbb{R}^n$ is a subspace if and only if it is the span of a set of vectors, i.e. $S = \text{span}\{\mathbf{u}_1, ..., \mathbf{u_m}\}$. If the collection $\{\mathbf{u}_1, ..., \mathbf{u_m}\}$ is independent, it will be a basis for S.

For the questions below, decide if the given sets are subspaces or not. If you are claiming that the set is a subspace, either show the definition holds or write S as a span of a set of vectors (better yet do both and give the dimension). If you are claiming that the set is not a subspace, then find vectors $\mathbf{u}$, $\mathbf{v}$ and numbers α and β such that $\mathbf{u}$ and $\mathbf{v}$ are in S but $\alpha\mathbf{u} + \beta\mathbf{v}$ is not. Also, every subspace must have the zero vector. If it is not there, the set is not a subspace.

Subspaces of $\mathbb{R}^2$

From the Theorem above, the only subspaces of $\mathbb{R}^n$ are: The set containing only the origin, the lines through the origin and $\mathbb{R}^2$ itself. Anything else is not.

1. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 5x + 3y = 0 \right\}$.
2. $S = \left\{ \begin{bmatrix} -3a \\ 5a \end{bmatrix} : a \text{ is any number} \right\}$.
3. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x + y = 3 \right\}$.
4. $S = \left\{ \begin{bmatrix} a \\ 3 - 2a \end{bmatrix} : a \text{ is any number} \right\}$.
5. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 = 0 \right\}$.
6. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 = 1 \right\}$.
7. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$.
8. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \text{ and } y \text{ are rational numbers} \right\}$.
9. $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \text{ and } y \text{ are any two numbers} \right\}$.

Subspaces of $\mathbb{R}^3$

From the Theorem above, the only subspaces of $\mathbb{R}^n$ are: The set containing only the origin, the lines through the origin, the planes through the origin and $\mathbb{R}^3$ itself. Anything else is not.

10. $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x + 3y + 2z = 0 \right\}$.
11. $S = \left\{ \begin{bmatrix} a - b \\ a + b \\ -4a + b \end{bmatrix} : a \text{ and } b \text{ are any two numbers} \right\}$.
12. $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + y - 7z = 3 \right\}$.

13. $S = \left\{ \begin{bmatrix} 4+2a+b \\ 2+3a-2b \\ 1+a \end{bmatrix} : a \text{ and } b \text{ are any two numbers} \right\}$.

14. $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2+y^2+z^2=0 \right\}$.

15. $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2+y^2+z^2=1 \right\}$.

16. $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : xyz \geq 0 \right\}$.

17. $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x+3y+2z=0 \text{ and } x-y+3z=0 \right\}$.

18. $S = \left\{ \begin{bmatrix} 2a \\ -3a \\ -4a \end{bmatrix} : a \text{ is any number} \right\}$.

19. $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x,y \text{ and } z \text{ are any three numbers} \right\}$.

Subspaces of $\mathbb{R}^n$

From the Theorem above, the only subspaces of $\mathbb{R}^n$ are spans of vectors. One way to describe a subspace would be to give a set of vectors which span it, or to give its basis. Questions 2, 11 and 18 do just that. Another way would be to describe the subspace as a solution set of a set of *homogeneous* equations. (Why homogeneous?) Compare Questions 1 and 3, or Questions 10 and 12.) Anything else is not a subspace.

20. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} : x_1+2x_2+3x_3+...+nx_n=0 \right\}$. If n is confusing, rewrite with $n=4$ or $n=5$.

21. $S = \left\{ \begin{bmatrix} a_1+a_2 \\ a_2+a_3 \\ a_3+a_4 \\ \cdot \\ \cdot \\ a_{n-2}+a_{n-1} \\ a_{n-1}+a_1 \end{bmatrix} : a_1, a_2, ..., a_{n-1} \text{ are any numbers} \right\}$. If n is confusing, rewrite with $n=4$ or $n=5$.

22. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : 2x_1+x_2-7x_3+8x_4=13 \right\}$.

23. $S = \left\{ \begin{bmatrix} 2a+b-3c \\ 4a-2b+5c \\ a \\ a+b+c \end{bmatrix} : a, b \text{ and } c \text{ are any three numbers} \right\}$.

24. $S = \left\{ \begin{bmatrix} 1+2a+b-3c \\ 5-4a-2b+5c \\ 3+a \\ 2-a+b+c \end{bmatrix} : a,\ b \text{ and } c \text{ are any three numbers} \right\}.$

25. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} : x_1^2 + x_2^2 + ... + x_n^2 = 0 \right\}.$

26. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} : x_1^2 + x_2^2 + ... + x_n^2 = 1 \right\}.$

27. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : 5x_1 + 3x_2 + 2x_3 + 9x_4 = 0 \text{ and } 2x_1 + 7x_2 - x_4 = 0 \right\}.$

28. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : 4x_1 + 2x_2 + x_3 + 8x_4 = 0 \text{ and } 3x_1 + 8x_3 - x_4 = 0 \text{ and } 2x_1 + x_2 + 5x_4 = 0 \right\}.$

29. $S = \left\{ \begin{bmatrix} 2a \\ -3a \\ -4a \\ 6a \\ a \end{bmatrix} : a \text{ is any number} \right\}.$

30. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} : x_1,x_2...x_n \text{ are any numbers} \right\}.$

Subspaces as ranges of transformations Since any subspace S is the span of a set of vectors, we can make a transformation by taking those vectors which span S and make them the columns of a matrix. In that way, every subspace is the range of a transformation. For example, the subspace in Question 23 is spanned by

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

the matrix is

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & -2 & 5 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

which defines a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$.

Exercise: Write down transformations with their matrices so that the subspaces in Questions 2, 11, 18, 21, 23 and 29 are the ranges of the transformations.

Subspaces as kernels of transformations Since any subspace S can be given as the solution set of a set of homogeneous equations, we can define a transformation using those equations and the subspace becomes the kernel. For example, the subspace in Question 28 is the kernel of the transformation given by

$$\begin{array}{cccc} T & : & \mathbb{R}^4 & \to & \mathbb{R}^3 \\ T & : & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} & \to & \begin{bmatrix} 4x_1 + 2x_2 + x_3 + 8x_4 \\ 3x_1 + 8x_3 - x_4 \\ 2x_1 + x_2 + 5x_4 \end{bmatrix} \end{array}$$

Exercise: Write down transformations with their matrices so that the subspaces in Questions 1, 10, 17, 20, 27 and 28 are the kernels of the transformations.

Solutions to the Questions

1. This is a subspace spanned by the single vector $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

2. This is a subspace spanned by the single vector $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

3. This is not a subspace. For example, the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set but the vector $2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not. Also, the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in the set.

4. This is not a subspace. For example, the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set (set $a = 1$) but the vector $2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not (if $a = 2$, then $3 - 2a \neq 2$). Also, the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in the set.

5. This is a subspace $S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

6. This is not a subspace because the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in the set.

7. This is not a subspace. For example, the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set, but the vector $-1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is not.

8.

9. This is not a subspace. For example, the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set, but the vector $\pi\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}$ is not.

10. This is a subspace. It is all of $\mathbb{R}^2$.

11. This is a subspace spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

12. This is a subspace spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

13. This is not a subspace because the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the set.

14. This is not a subspace because the zero vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is not in the set. If you try to solve for a, b, c setting $\begin{bmatrix}4+2a+b\\2+3a-2b\\1+a\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ you'll see it is inconsistent.

15. This is a subspace with a single vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

16. This is not a subspace because the zero vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is not in the set.

17.

18. This is not a subspace. For example, the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is in the set, but the vector $-1\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$ is not.

19. This is a subspace spanned by the vector $\begin{bmatrix}-11\\13\end{bmatrix}$.

20. This is a subspace spanned by the vector $\begin{bmatrix}2\\-34\end{bmatrix}$.

21. This is a subspace. It is all of $\mathbb{R}^3$.

22. This is a subspace just like Questions 1 and 10. Its dimension is $n-1$. It is called a hyperplane. It is spanned by the $n-1$ independent vectors

$$\begin{bmatrix}-2\\1\\0\\0\\0\\.\\.\\.\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\\0\\.\\.\\.\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\0\\1\\0\\.\\.\\.\\0\end{bmatrix}, \ldots, \begin{bmatrix}-n\\0\\0\\.\\.\\.\\.\\0\\1\end{bmatrix}$$

23. This is a subspace spanned by the vectors

$$\begin{bmatrix}1\\0\\0\\.\\.\\.\\.\\.\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\.\\.\\.\\.\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\\.\\.\\.\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\0\\.\\.\\.\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\.\\.\\.\\.\\0\\1\\1\end{bmatrix}$$

24. This is not a subspace because the zero vector $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ is not in the set.

25. This is a vector subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \\ 1 \end{bmatrix}.$$

26. This is not a subspace because the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the set. Try it.

27. This is the subspace containing only the zero vector.

28. This is not a subspace because the zero vector is not in the set

29. This is a vector subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} -14 \\ 4 \\ 29 \\ 0 \end{bmatrix}, \begin{bmatrix} -66 \\ 23 \\ 0 \\ 29 \end{bmatrix}.$$

30. This is a vector subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} -5 \\ 5 \\ 2 \\ 1 \end{bmatrix}.$$

31. This is a vector subspace of $\mathbb{R}^5$ spanned by

$$\begin{bmatrix} 2 \\ -3 \\ -4 \\ 6//1 \end{bmatrix}.$$

32. This is a subspace. It is all of $\mathbb{R}^n$.