Spring 2019 Math 300 Midterm exam

Write clearly and legibly. Justify all your answers.

You will be graded for correctness and clarity of your solutions.

You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.

You can use elementary algebra and any result that we proved in class (but not in the homework). You need to prove everything else.

Please raise your hand and ask a question if anything is not clear.

This exam contains 7 pages please make sure you have a complete exam. You have 50 minutes. Good luck

NAME:__________

NAME:_________

PROBLEM 1 __10__ pt

PROBLEM 2 __10__ pt

PROBLEM 3 __5__ pt

PROBLEM 4 __10__ pt

PROBLEM 5 __10__ pt

Total __45__ pt

- **Problem 1** Recall that the Fibonacci sequence is defined by:

  $u_1 = 1$

  $u_2 = 1$

  $u_{n+1} = u_n + u_{n-1}$ if $n + 1 \geq 3$

  Prove that $\sum_{i=1}^{n} u_i = u_{n+2} - 1$

Proof by induction

Base case: if $n = 1$ $\quad u_1 = 1$, $\quad u_3 - 1 = 2 - 1 = 1$

No need to consider n=2

Induction step: assume $\sum_{i=1}^{k} u_i = u_{k+2} - 1$ for some $k \geq 1$

$u_{k+2} + u_{k+1} = u_{k+3}$ by Fibonacci definition

then $\sum_{i=1}^{k+1} u_i = \underbrace{\sum_{i=1}^{k} u_i}_{\text{by induction assumption}} + u_{k+1} = u_{k+2} - 1 + u_{k+1} = u_{k+3} - 1 = u_{(k+1)+2} - 2$

- **Problem 2** Let A_i be the set of all positive integers divisible by i, that is $A_i = \{n \in Z^+ | i \text{ divides } n\}$.

  1. Prove that $A_2 \cap A_3 = A_6$

First we'll show $A_2 \cap A_3 \subseteq A_6$: if $x \in A_2 \cap A_3$ then $x \in A_2$ so 2 div x so $x = 2h$ for some $h \in \mathbb{Z}$, and $x \in A_3$ so 3 div x so $x = 3k$ for some $k \in \mathbb{Z}$; since x is even and 3 is odd, k must be even, so $k = 2\ell$ for some ℓ in $\mathbb{Z}$ and $x = 3 \cdot 2\ell$ so 6 div x and $x \in A_6$

Now we'll show $A_6 \subseteq A_2 \cap A_3$: if $x \in A_6$ then 6 div x so $x = 6m$ for some m in $\mathbb{Z}$, therefore $x = 2(3m)$ so 2 div x and $x \in A_2$ and $x = 3(2m)$ so 3 div x and $x \in A_3$. Since $x \in A_2$ and $x \in A_3$ then $x \in A_2 \cap A_3$

  2. Prove that for all $k \geq 2$ the set $\bigcap_{i=2}^{k} A_i$ is infinite.

Consider $n \cdot k! = n \cdot (1 \cdot 2 \cdot 3 \cdot \cdots \cdot k)$ then for any $n \in \mathbb{Z}^+$

$n k! \in \bigcap_{i=2}^{k} A_i$

In general $A_m \cap A_n \neq A_{m \cdot n}$ for example
$12 \in A_6 \cap A_4$ but $12 \notin A_{24}$

  3. Is $\bigcap_{i=2}^{\infty} A_i$ infinite ? Justify your answer.

No $\bigcap_{i=2}^{\infty} A_i = \emptyset$ since for any $n \in \mathbb{Z}^+$ $x \notin A_{n+1}$

- **Problem 3** The exclusive or $P \oplus Q$ is defined by the following table:

P	Q	$P \oplus Q$
1	1	0
1	0	1
0	1	1
0	0	0

That is $P \oplus Q$ is true exactly when one of P or Q is True and the other is False.

Write a statement equivalent to $P \oplus Q$ that only uses P, Q and the connectives $\neg$, $\vee$. No justification necessary.

$P \oplus Q$ equiv $(P \wedge \neg Q) \vee (\neg P \wedge Q)$ equiv $\neg(\neg P \vee Q) \vee \neg(P \vee \neg Q)$

or

$P \oplus Q$ equiv $(P \vee Q) \wedge \neg(P \wedge Q)$ equiv $(P \vee Q) \wedge (\neg P \vee \neg Q)$

equiv $\neg\big(\neg(P \vee Q) \vee \neg(\neg P \vee \neg Q)\big)$

- **Problem 4**

  Define a function $f : \mathrm{Z} \to \mathrm{Z}$ by:

$$f(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x+5 & \text{if } x < 0 \end{cases}$$

  1. Is f injective ? *(Give a proof).*

No $f(0) = f(-3)$

  2. Is f surjective ? *(Give a proof).*

yes given $y \in \mathbb{Z}$ if $y \geq 2$ take $x = y-2$ then $x \geq 0$
and $f(x) = x+2 = y$
if $y < 2$ take $x = y-5$ then $x < 0$ and $f(x) = x+5 = y$

- **Problem 5** For each statement below circle if it is true or false and give a proof. $EVEN^+$ is the set of even positive integers.

1. $\forall y \in EVEN^+ \, \exists x \in Z \quad y = \frac{x^2}{2}$ TRUE FALSE

Take $y = 4$ then there is no x s.t $4 = \frac{x^2}{2}$
since this is equivalent to $8 = x^2$ and if
$x = \pm 1$ $x^2 = 1$, if $x = \pm 2$ $x^2 = 4$, if $x \geq 3$ or $x \leq -3$ $x^2 \geq 9$
so there is no x in Z with $x^2 = 8$

2. $\forall x \in EVEN^+ \, \exists y \in Z \quad y = \frac{x^2}{2}$ TRUE FALSE

Given x in $EVEN^+$, since
is even then $x = 2k$ for some k in Z
and therefore $\frac{x^2}{2} = \frac{4k^2}{2} = 2k^2$ is in Z
so we can take $y = \frac{x^2}{2}$

3. $\exists x \in Z \, \forall y \in Z \quad y + x > 3$ TRUE FALSE

Given x take $y = -x$ then $y + x = 0$ which is
less than 3