SPRING 2018 MATH 300 B FINAL EXAM

Write clearly and legibly. Justify all your answers.

You will be graded for correctness and clarity of your solutions.

You may use one 8.5 x 11 sheet of notes; writing is allowed on both sides. You may use a calculator.

You can use elementary algebra and any result that we proved in class (but not in the homework). You need to prove everything else.

Please raise your hand and ask a question if anything is not clear.

This exam contains 8 pages, please make sure you have a complete exam. You have 1 hr and 50 minutes. Good luck

NAME:_________

PROBLEM 1 _______

PROBLEM 2 _______

PROBLEM 3 ________

PROBLEM 4 ________

PROBLEM 5 ________

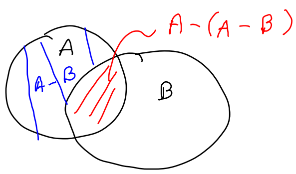
PROBLEM 6 ________

PROBLEM 7 ________

Total ____________

- **Problem 1** Given sets A, B, prove that $(A - (A - B)) \subseteq B$. Give an example to show that equality does not have to hold.

Assume $x \in A-(A-B)$ then $x \in A$ and $x \notin A-B$; $x \notin A-B$ if either $x \notin A$, and this is not the case or $x \in B$; so x must be in B.



so $A \cap B \neq B$

if $A = \{1\}$ and $B = \{1, 2\}$ $\quad A - B = \emptyset \quad A-(A-B) = \{1\}$

so $A-(A-B) \neq B$

- **Problem 2** Write a statement equivalent to the negation of

$$\exists x \in A\, \forall y \in B\, (x \leq y) \Rightarrow (\exists z \in C\, ((z > x) \Rightarrow (z > y \wedge z = y)))$$

that does not use the negation symbol $\neg$. You are allowed to use $\neq$.

$\forall x \in A \;\; \exists y \in B \;\;\; x \leq y \;\wedge\; \forall z \in C \;\;\; x > y \;\wedge\; (z \leq y \;\vee\; z \neq y))$

- **Problem 3** Prove that the sum of two odd perfect squares is never a perfect square. A perfect square is an integer z such that $z = k^2$ for some integer k.

Assume $x = a^2$ and $y = b^2$ are two odd perfect squares a and b have to be odd as well. Let $a = 2k+1$ and $b = 2h+1$ for some h, k in $\mathbb{Z}$. Then $x + y = a^2 + b^2 = (2k+1)^2 + (2h+1)^2 = 4(k^2+h^2+k+h) + 2$ is congruent to 2 mod 4

We shall show that no perfect square is congruent to 2 mod 4, therefore $x+y$ is not a perfect square:

Let $z = m^2$ a perfect square: m can be congruent to $0,1,2,3$ mod 4, and m^2 is then congruent to $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 0$, $3^2 \equiv 1$ mod 4 so m^2 is never congruent to 2 mod 4

1 2 3 4 5 6
0 1 5 3 4 8

- **Problem 4** Define a function $f : Z \to Z$ by:

$$f(x) = \begin{cases} x+2 & \text{if 3 divides } x \\ x-1 & \text{otherwise} \end{cases}$$

1. Is f injective ? *(Give a proof).*

yes : if $x_1 \neq x_2$ and both x_1 and x_2 are divisible by 3, then clearly $x_1+2 \neq x_2+2$; if neither x_1 and x_2 are divisible by 3, then clearly $x_1-1 \neq x_2-1$; if one of them, say x_1 is divisible by 3 and x_2 is not then x_2 is congruent to 1 or 2 mod 3 and x_2-1 is congruent to 0 or 1, while x_1+2 is congruent to 2 mod 3 therefore $x_2-1 \neq x_1+2$

2. Is f surjective ? *(Give a proof).*

yes if $y \equiv 2 \bmod 3$ take $x = y-2$ then 3 div x and $f(x) = x+2 = y$
if $y \equiv 1 \bmod 3$ take $x = y+1$ then 3 does not divide x and $f(x) = x-1 = y$
if $y \equiv 0 \bmod 3$ take $x = y+1$ then 3 does not divide x and
$f(x) = x-1 = y$

3. Prove that $\forall n \in Z^+ \forall m \in Z f^{3n}(m) = m$ [here $f^n(x)$ means ....]

If $n = 1$ if $x \equiv 0 \bmod 3$ $f^3(x) = f(f(x+2)) = f(x+1) = x$
if $x \equiv 1 \bmod 3$ $f^3(x) = f(f(x-1)) = f(x+1) = x$
if $x \equiv 2 \bmod 3$ $f^3(x) = f(f(x-1)) = f(x-2) = x$
(since now 3 divides $x-2$)

Induction step: assume $f^{3k}(x) = x$ then $f^{3(k+1)}(x) =$
$= f^{3k+3}(x) = f^{3k}(f^3(x)) = f^{3k}(x) = x$

- **Problem 5** Find all integer solutions of $3 \cdot 7^{1022} x \equiv 25 \bmod 31$

31 is prime so $7^{30} \equiv 1 \mod 31$ $\quad 1022 = 30 \times 34 + 2$

$3 \cdot 7^2 (7^{30})^{34} \equiv 25 \mod 31$

$3 \cdot 49 X \equiv 25 \mod 31$

$147 X \equiv 25 \mod 31$

$147 = 31 \times 4 + 23$

$31 = 23 \times 1 + 8$

$23 = 8 \times 2 + 7$

$8 = 7 \times 1 + 1$

$7 = 1 \times 7 + 0$

$1 = 31 \cdot 3 + (147 - 31 \cdot 4)(-4) = 147(-4) + 31(19)$

$1 = 23(-1) + (31 - 23)3 = 31 \cdot 3 + 23(-4)$

$1 = 8 - (23 - 8 \times 2) = 23(-1) + 8 \times 3$

$1 = 8 - 7$

so $25 = 147(-100) + 31 \cdot (19 \cdot 25)$

so $X = -100 + 31K$

- **Problem 6** Show that the relation r defined on R, the set of real numbers by xry iff $x - y \in Z$ is an equivalence relation.

1) $\forall x \in Z \quad x - x = 0 \in Z \quad$ so $x r x$

2) $\forall x, y \in Z \quad x - y \in Z \Rightarrow y - x = -(x - y) \in Z \quad$ so $xry \Rightarrow yrx$

3) $\forall x, y, z \in Z \quad x - y \in Z \wedge y - z \in Z \Rightarrow (x - y) + (y - z) = x - z \in Z$

so $xry \wedge yrz \Rightarrow xrz$

- **Problem 7** Prove that if A and B are ~~denumerable sets,~~ then their cartesian product $A \times B$ is denumerable. You can assume ...

Let $f \; \mathbb{Z}^+ \to A$ and $g \; \mathbb{Z}^+ \to B$ be bijections

Then $h \; \mathbb{Z}^+ \times \mathbb{Z}^+ \to A \times B$
$(x, y) \to (f(x)\; g(x))$

is a bijection since $k(x) = A \times B \to \mathbb{Z}^+ \times \mathbb{Z}^+$
$k(a, b) = (f^{-1}(a), g^{-1}(b))$

is such that $h(k(a,b)) = h(f^{-1}(a), g^{-1}(b)) = (a,b)$

and $k(h(x,y)) = k(f(x), g(y)) = (x, y)$

so $k = h^{-1}$ and h is a bijection