The non-random nature of random limit points

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Theorem 1 *If for a sequence* $\{Z_t,\ t = 1, 2, \ldots\}$ *of random variables on a complete probability space* $(\Omega, \mathcal{F}, P)$ *with values in a compact metric space* (E, d) *we have*

$$P\{Z_t \in V \ \ i.\ o.\} = 0 \ \text{ or } 1 \ \text{ for every open set } V, \tag{1}$$

and if

$$G = \{x : P\{d(Z_t, x) < \varepsilon \ \ i.\ \ o.\ \} = 1,\ \forall\, \varepsilon > 0\},$$

then $G \neq \emptyset$ *and*

$$P\{\omega : A(\omega) = G\} = 1, \tag{2}$$

where

$$\begin{aligned} A(\omega) &= \{x : \liminf_{t\to\infty} d(Z_t(\omega), x) = 0\} \\ &= \bigcap_{n=1}^{\infty} \overline{\{Z_t(\omega) : t \geq n\}}. \end{aligned}$$

1 Proof.

First G is closed. For if G is non-empty and if y is a limit point of G which we may suppose is not isolated, then for any neighborhood V of y we can find a neighborhood V_0 of some point of G which is contained in V. But then $P\{Z_t \in V \text{ i. o.}\} \geq P\{Z_t \in V_0 \text{ i. o.}\} = 1$. Since V is any neighborhood of y the result follows.

Next we show G is non-empty. Without loss of generality we may suppose that the diameter of E is 1. Let $N(x, r)$ denote the open ball centered at x with radius r. Since E is compact, we can cover E with a finite number of balls of radius 1/2, and since Z_t stays in E for all t, we can find at least one of these balls of radius 1/2 that will be entered infinitely often with positive probability and therefore with probability 1 by (1). Call this ball $V_1 = N(z_1, 1/2)$. Cover the compact set $\overline{V_1}$ with a finite number of balls of radius 1/4 each having a non-empty intersection with V_1. Then Z_t must enter at least one of these, i. o., with probability 1. Write it as $V_2 = N(z_2, 1/4)$ where $z_2 \in \overline{V_2}$. We continue inductively. At the kth stage we get a ball of radius $1/2^k$ of the form $V_k = N(z_k, 1/2^k)$ where $z_k \in \overline{V_{k-1}}$. Also at each stage we can find a null set of paths, say Ξ_k, such that if $\omega \notin \Xi_k$, then $Z_t(\omega) \in V_k$ i. o.. The sequence of centers of these balls, $\{z_k\}$, clearly form a Cauchy sequence. Let $z = \lim_k z_k$. If $r > 0$ then $V_k \subset N(z, r)$ for all k sufficiently large. If $\omega \notin \cup_k \Xi_k$, then $Z_t(\omega)$ will enter all V_k infinitely often and therefore will enter $N(z, r)$ infinitely often. In as much as $\cup_k \Xi_k$ is a null set, z must be in G by definition of G.

Let $z_1^n, \dots, z_{k(n)}^n$ be points in G such that every point of G is at a distance strictly less than $1/2^n$ of one of these points. This can be done for every n because G is compact. Also put

$$\mathcal{O}_n = \bigcup_{i=1}^{k_n} N(z_i^n, \frac{1}{2^n}).$$

Then $G \subset \mathcal{O}_n$ for every n; moreover $\min\left(d(x, y) : x \in G,\ y \in \mathcal{O}_n^c\right) > 0$. By definition of G for each $y \in \mathcal{O}_n^c$ there is a $r = r(y)$ such that $P\{Z_t \in N(y, r) \text{ i. o.}\} = 0$, or, put another way,

$$P\{Z_t \notin N(y, r) \text{ for all } t \geq \ell \text{ for some } \ell\} = 1$$

The sets $\{N(y, r(y))\}_{y \in \mathcal{O}_n^c}$ make a covering of $\mathcal{O}_n^c$, so there exists $y_1^n, \dots, y_{m_n}^n$ in $\mathcal{O}_n^c$ such that

$$\mathcal{O}_n^c \subset \bigcup_{i=1}^{m_n} N(y_i^n, r_i^n), \quad r_i^n = r(y_i^n).$$

Define

$$\begin{aligned} \Omega_n &= \left\{w : Z_t(\omega) \in N(z_i^n,\, 2^{-n}) \text{ i. o. } \forall\, i = 1, \dots, k_n\right\} \\ \Lambda_\ell &= \{\omega : Z_t(\omega) \notin N(y_i^n, r_i^n)\ \forall\, t \geq \ell\ \ \forall\, i = 1, \dots, m_n\} \quad \Lambda = \bigcup_{\ell=1}^{\infty} \Lambda_\ell \end{aligned}$$

Then $P(\Omega_n \cap \Lambda) = 1$ for every n, hence, if we put

$$\Omega_\infty = \bigcap_{n \geq 1} \Omega_n \Lambda,$$

then $P(\Omega_\infty) = 1$. Finally let us define

$$\widetilde{\Omega}_r = \{w : Z_t(w) \in N(z,r) \text{ i. o. for every } z \in G\}.$$

Clearly $\Omega_\infty \subset \widetilde{\Omega}_r$ for every $r > 0$. Indeed $\Omega_n \subset \widetilde{\Omega}_r$ as soon as $1/2^n < r/2$. It follows that $P(\widetilde{\Omega}_{0+}) = 1$ where

$$\widetilde{\Omega}_{0+} = \bigcap_{r>0,\, r \text{ rational}} \widetilde{\Omega}_r.$$

Recalling the definition of $A(\omega)$ it is now clear that

$$\begin{aligned} \omega \in \Omega_\infty \quad &\Rightarrow \quad A(\omega) \subset G \\ \omega \in \widetilde{\Omega}_{0+} \quad &\Rightarrow \quad G \subset A(\omega). \end{aligned}$$

These implications and $P\{\Omega_\infty \bigtriangleup \widetilde{\Omega}_{0+}\} = 0$ yield (2).

For a proof of a similar result for normalized sums of independent random variables, see Theorem 1 on page 1174 of the paper by H. Kesten, (970), The limit points of normalized random walk. *Ann. Math. Statist.* **41**.