L'Hôpital's Rule

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Often one is faced with the evaluation of limits of quotients $f(x)/g(x)$ where f and g both tend to zero or infinity. The collection of related results that go under the name of "l'Hôpital's rule" enable one to evaluate such limits in many cases by examining the quotient of the derivatives, $f'(x)/g'(x)$. These results are all applications of the generalized mean-value theorem (Theorem 5.12 in Apostol).

The cases involving the indeterminate form $0/0$ can be summarized as follows.

Theorem 1 (L'Hôpital's Rule I)**.** *Suppose f and g are differentiable functions on (a, b) and*

$$\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = 0.$$

If g' never vanishes on (a, b) and the limit

$$\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L$$

exists, then g never vanishes on (a, b) and

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = L.$$

The same result holds for

- *the left-hand limit $\lim_{x\to a-}$, if f and g are differentiable on an interval (d, a),*
- *the two-sided limit $\lim_{x\to a}$, if f and g are differentiable on intervals (d, a) and (a, b), and*
- *the limit $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$, if f and g are differentiable on an interval (b, ∞) or $(-\infty, b)$.*

Proof. If we (re)define $f(a)$ and $g(a)$ to be 0, then f and g are continuous on the interval $[a, x]$ for $x < b$. By the ordinary mean value theorem, if $x \in (a, b)$ we have $g(x) = g(x) - g(a) = g'(c)(x-a)$ for some $c \in (a, x)$, and $g'(c) \neq 0$ by assumption, so $g(x) \neq 0$. By the generalized mean-value theorem, for each $x \in (a, b)$ there exists $c \in (a, x)$ (depending on x) such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Since $c \in (a, x)$, c approaches $a+$ as x does, so

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{c\to a+} \frac{f'(c)}{g'(c)} = L.$$

The proof for left-hand limits is similar, and the case of two-sided limits is obtained by combining right-hand and left-hand limits. Finally, for the case $a = \pm\infty$, we set $y = 1/x$ and consider the functions $F(y) = f(1/y)$ and $G(y) = g(1/y)$. Since $F'(y) = -f'(1/y)/y^2$

and $G'(y) = -g'(1/y)/y^2$, we have $F'(y)/G'(y) = f'(1/y)/g'(1/y)$, so by the results just proved,

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \lim_{y\to 0\pm} \frac{F(y)}{G(y)} = \lim_{y\to 0\pm} \frac{F'(y)}{G'(y)} = \lim_{x\to\pm\infty} \frac{f'(x)}{g'(x)}.$$

∎

Under the conditions of this theorem, it may well happen that $f'(x)$ and $g'(x)$ tend to zero also, so that the limit of $f'(x)/g'(x)$ cannot be evaluated immediately. In this case we can apply the theorem again to evaluate the limit by examining $f''(x)/g''(x)$. More generally, if the functions $f, f', \ldots, f^{(k-1)}, g, g', \ldots, g^{(k-1)}$ all tend to zero as x tends to $a+$ or $a-$ or $\pm\infty$, but $f^{(k)}(x)/g^{(k)}(x) \to L$, then $f(x)/g(x) \to L$.

Example. Let $f(x) = 2x - \sin 2x$, $g(x) = x^2 \sin x$, $a = 0$. Then f, g, and their first two derivatives vanish at $x = a$, but the third derivatives do not, so

$$\begin{aligned}\lim_{x\to 0} \frac{2x - \sin 2x}{x^2 \sin x} &= \lim_{x\to 0} \frac{2 - 2\cos 2x}{2x\sin x + x^2 \cos x} = \lim_{x\to 0} \frac{4\sin 2x}{(2 - x^2)\sin x + 4x\cos x} \\ &= \lim_{x\to 0} \frac{8\cos 2x}{(6 - x^2)\cos x - 6x\sin x} = \frac{4}{3}.\end{aligned}$$

The corresponding result for limits of the form ∞/∞ is also true.

Theorem 2 (L'Hôpital's Rule II)**.** *Theorem 1 remains valid when the hypothesis that* $\lim f(x) = \lim g(x) = 0$ *(as* $x \to a+$, $x \to a-$, *etc.) is replaced by the hypothesis that* $\lim |f(x)| = \lim |g(x)| = \infty$.

Proof. We consider the case of left-hand limits as $x \to a-$; the other cases follow as in Theorem 1.

Given $\epsilon > 0$, we wish to show that $\big|[f(x)/g(x)] - L\big| < \epsilon$ provided that x is sufficiently close to a on the left. Since $f'(x)/g'(x) \to L$ and $|g(x)| \to \infty$, we can choose $x_0 < a$ so that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2} \text{ and } g(x) \neq 0 \text{ for } x_0 < x < a.$$

Moreover, by the generalized mean value theorem, if $x_0 < x < a$ we have

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)} \text{ for some } c \in (x_0, x),$$

and hence, since $x_0 < c < a$,

$$\left|\frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L\right| < \frac{\epsilon}{2} \text{ for } x_0 < x < a. \tag{*}$$

Next, we have

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(x_0)}{g(x)}}{1-\dfrac{g(x_0)}{g(x)}},$$

by dividing top and bottom by $g(x)$. Solving this equation for the fraction $f(x)/g(x)$ yields

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} - \frac{f(x)-f(x_0)}{g(x)-g(x_0)}\frac{g(x_0)}{g(x)} + \frac{f(x_0)}{g(x)}. \tag{**}$$

Since $|g(x)| \to \infty$ as $x \to a$, the quotients $f(x_0)/g(x)$ and $g(x_0)/g(x)$ tend to zero as $x \to a$, and by (*) the difference quotient remains bounded, so the second two terms on the right of (**) tend to zero. Hence, for x sufficiently close to a we have

$$\left|\frac{f(x)}{g(x)} - \frac{f(x)-f(x_0)}{g(x)-g(x_0)}\right| < \frac{\epsilon}{2}.$$

Combining this with (*), we obtain

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon,$$

which is what we needed to show. ∎

The following special cases of Theorem 2 are of fundamental importance.

Corollary 1. *For any $a > 0$ we have*

$$\lim_{x\to+\infty}\frac{x^a}{e^x} = \lim_{x\to+\infty}\frac{\log x}{x^a} = \lim_{x\to 0+}\frac{\log x}{x^{-a}} = 0.$$

That is, the exponential function e^x grows more rapidly than any power of x as $x \to +\infty$, whereas $|\log x|$ grows more slowly than any positive power of x as $x \to +\infty$ and more slowly than any negative power of x as $x \to 0+$.

Proof. For the first limit, let k be the smallest integer that is $\geq a$. A k-fold application of Theorem 2 yields

$$\lim_{x\to+\infty}\frac{x^a}{e^x} = \lim_{x\to+\infty}\frac{a(a-1)\cdots(a-k+1)x^{a-k}}{e^x},$$

and the latter limit is zero because $a-k \leq 0$. For the other two limits, a single application of Theorem 2 suffices:

$$\lim_{x\to+\infty}\frac{\log x}{x^a} = \lim_{x\to+\infty}\frac{1}{ax^a} = 0, \qquad \lim_{x\to 0+}\frac{\log x}{x^{-a}} = \lim_{x\to 0+}\frac{x^a}{a} = 0.$$

∎

By raising the quantities in Corollary 1 to a positive power b and replacing a by a/b, we obtain a more general result:

Corollary 2. *For $a, b > 0$ we have* $\displaystyle\lim_{x\to+\infty}\frac{x^a}{e^{bx}} = \lim_{x\to+\infty}\frac{(\log x)^b}{x^a} = \lim_{x\to 0+}\frac{|\log x|^b}{x^{-a}} = 0.$