These are a few exercises to give you some practice in thinking about linear transformations on spaces of functions. In each case the inner product is given by $\langle f, g\rangle = \int_I f^*(x)g(x)\,dx$ where I is the interval on which the functions in question are defined. Do the first two for discussion on Monday 4/14, and **hand in** the last two on Wednesday 4/16.

(1) Let $V = L^2[-1, 1]$. Define $T : V \to V$ by $(Tf)(x) = \frac{1}{2}[f(x) + f(-x)]$. Show that T is an orthogonal projection. (You have to check that T is Hermitian and that $T^2 = T$.) Can you describe the space that T projects onto, i.e., the range of T? (A few words should suffice.)

(2) Let $V = L^2[-1, 1]$. Define $T : V \to V$ by $Tf(x) = (x^2 + 1)e^{ix}f(x)$.
   a. Compute the inverse transformation T^{-1}.
   b. Compute the adjoint transformation T^*, i.e, the transformation such that $\langle Tf, g\rangle = \langle f, T^*g\rangle$. (Hint: The complex conjugate of e^{ix} is e^{-ix}.)

(3) Let $V = L^2(\mathbb{R})$. Define $T : V \to V$ by $(Tf)(x) = f(7x - 2)$. Compute the inverse and adjoint of T as in the preceding problem. (They're closely related to each other, and both have a form similar to T itself.)

(4) Let V be the space of all infinitely differentiable functions f on $[0, 2\pi]$ such that $f^{(k)}(0) = f^{(k)}(2\pi)$ for all $k \geq 0$. (This is really the space of all infinitely differentiable functions on a circle, where the circle is parametrized by an angular variable.) Define $T : V \to V$ by $(Tf)(x) = (1/i)f'(x)$.
   a. Show that T is Hermitian.
   b. Find the eigenvectors and eigenvalues of T.