Norms

A norm is a way of measuring the length of a vector. Let V be a vector space. A *norm* on V is a function $\|\cdot\| : V \to [0,\infty)$ satisfying

(i) $(\forall\, v \in V)\ \|v\| \geq 0$, and $\|v\| = 0$ iff $v = 0$

(ii) $(\forall\, \alpha \in \mathbb{F})(\forall\, v \in V)\ \|\alpha v\| = |\alpha| \cdot \|v\|$, and

(iii) (triangle inequality) $(\forall\, v, w \in V)\ \|v + w\| \leq \|v\| + \|w\|$.

The pair $(V, \|\cdot\|)$ is called a *normed linear space* (or normed vector space).

Fact. A norm $\|\cdot\|$ on a vector space V induces a metric d on V by

$$d(v,w) = \|v - w\|.$$

Exercise. Show that d is a metric on V.

All topological properties (e.g. open sets, closed sets, convergence of sequences, continuity of functions, compactness, etc.) will refer to those of the metric space (V, d).

Examples.

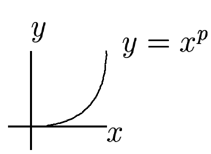
(1) ℓ^p norm on $\mathbb{F}^n$ $(1 \leq p \leq \infty)$

(a) $p = \infty$: $\quad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|,\ x \in \mathbb{F}^n$

(b) $1 \leq p < \infty$: $\quad \|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}},\ x \in \mathbb{F}^n$.
The triangle inequality

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$$

is known as "Minkowski's inequality." It is a consequence of Hölder's inequality. Integral versions of these inequalities are proved in real analysis texts, e.g., Folland or Royden. The proofs for vectors in $\mathbb{F}^n$ are analogous to the proofs for integrals



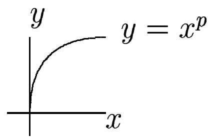
$(1 \leq p < \infty)$

Related observation: for $1 \leq p < \infty$, the map $x \mapsto x^p$ for $x \geq 0$ is convex.

(c) $0 < p < 1$: $\quad \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ is *not* a norm on $\mathbb{F}^n$. If $x = e_1$, and $y = e_2$,

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} = 2^{\frac{1}{p}} > 2 = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}},$$

so the triangle inequality does not hold.



$(0 < p < 1)$

Related observation: for $0 < p < 1$, the map $x \mapsto x^p$ for $x \geq 0$ is *not* convex.

(2) ℓ^p norm on ℓ^p (subspace of $\mathbb{F}^\infty$) $(1 \leq p \leq \infty)$

(a) $p = \infty$: $\ell^\infty = \{x \in \mathbb{F}^\infty : \sup_{i \geq 1} |x_i| < \infty\}$, $\|x\|_\infty = \sup_{i \geq 1} |x_i|$ for $x \in \ell^\infty$.

(b) $1 \leq p < \infty$: $\ell^p = \left\{x \in \mathbb{F}^\infty : (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}} < \infty\right\}$, $\|x\|_p = (\sum_{i=1}^\infty |x|^p)^{\frac{1}{p}}$ for $x \in \ell^p$. *Exercise.* Show that the triangle inequality follows from the finite-dimensional case.

(3) L^p norm on $C([a,b])$ $(1 \leq p \leq \infty)$

(a) $p = \infty$: $\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|$.
Since $|f(x)|$ is a continuous, real-valued function on the compact set $[a,b]$, it takes on its maximum, so the "sup" is actually a "max" here:

$$\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|.$$

(b) $1 \leq p < \infty$: $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$.
Use continuity of f to show that $\|f\|_p = 0 \Rightarrow f(x) \equiv 0$ on $[a,b]$. The triangle inequality

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}$$

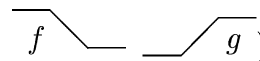
is Minkowski's inequality, a consequence of Hölder's inequality.

(c) $0 < p < 1$: $\left(\int_a^b |f(x)^p dx\right)^{\frac{1}{p}}$ is *not* a norm on $C([a,b])$.
"Pseudo-example": Let $a = 0$, $b = 1$,

$$f(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ 0 & \frac{1}{2} < x \leq 1 \end{cases} \quad \text{and } g(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1. \end{cases} .$$

Then

$$\begin{aligned} \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_0^1 |g(x)|^p dx\right)^{\frac{1}{p}} &= \left(\frac{1}{2}\right)^{\frac{1}{p}} + \left(\frac{1}{2}\right)^{\frac{1}{p}} \\ &= 2^{1-\frac{1}{p}} \\ &< 1 \\ &= \left(\int_0^1 |f(x)+g(x)|^p dx\right)^{\frac{1}{p}}, \end{aligned}$$

so the triangle inequality fails. Here, f and g are not continuous. *Exercise.* Adjust these f and g to be continuous (e.g., 

) to construct a legitimate counterexample to the triangle inequality.

Remark. There is also a Minkowski inequality for integrals: if $1 \le p < \infty$ and $u \in C([a,b] \times [c,d])$, then

$$\left(\int_a^b \left| \int_c^d u(x,y) dy \right|^p dx \right)^{\frac{1}{p}} \le \int_c^d \left(\int_a^b |u(x,y)|^p \, dx \right)^{\frac{1}{p}} dy \; .$$

Continuous Linear Operators on Normed Linear Spaces

Theorem. *Suppose* $(V, \|\cdot\|_V)$ *and* $W, \|\cdot\|_W)$ *are normed linear spaces, and* $L : V \to W$ *is a linear transformation. Then the following are equivalent:*

(a) L *is continuous.*

(b) L *is uniformly continuous.*

(c) $(\exists\, C)$ *so that* $(\forall\, v \in V)$ $\|Lv\|_W \le C\|v\|_V$.

Proof. (a) $\Rightarrow$ (c): Suppose L is continuous. Then L is continuous at $v = 0$. Let $\epsilon = 1$. Then $\exists\, \delta > 0$ so that if $\|v\|_V \le \delta$, then $\|Lv\|_W \le 1$ (as $L(0) = 0$). For any $v \ne 0$, $\left\| \frac{\delta}{\|v\|_V} v \right\|_V \le \delta$, so $\left\| L\left(\frac{\delta}{\|v\|_V} v \right) \right\|_W \le 1$, i.e., $\|Lv\|_W \le \frac{1}{\delta}\|v\|_V$. Let $C = \frac{1}{\delta}$.
(c) $\Rightarrow$ (b): Suppose $(\forall\, v \in V)$ $\|Lv\|_W \le C\|v\|_V$. Then $(\forall\, v_1, v_2 \in V)$ $\|Lv_1 - Lv_2\|_W = \|L(v_1 - v_2)\|_W \le C\|v_1 - v_2\|_V$. Hence L is uniformly continuous (given ϵ, let $\delta = \frac{\epsilon}{C}$, etc.). In fact, L is uniformly Lipschitz continuous with Lipschitz constant C.
(b)$\Rightarrow$(a) is immediate.

Definition. If $L : V \to W$ is a linear operator (where V and W are normed linear spaces), and $\sup_{v \in V, v \ne 0} \frac{\|Lv\|_W}{\|v\|_V} < \infty$, then L is called a *bounded linear operator* from V to W.

Remarks.

(1) Note that it is the *norm ratio* $\frac{\|Lv\|_W}{\|v\|_V}$ (or "stretching factor") that is bounded, *not* $\{\|Lv\|_W : v \in V\}$.

Exercise. Show that if $(\exists\, K)$ $(\forall\, v \in V)\|Lv\|_W \le K$, then $L \equiv 0$.

(2) The theorem above says that if V and W are normed linear spaces and $L : V \to W$ is linear, then L is continuous $\Leftrightarrow$ L is uniformly continuous $\Leftrightarrow$ L is a bounded linear operator.

Definition. If V and W are normed linear spaces and $L : V \to W$ is a bounded linear operator, define the *operator norm* of L to be

$$\|L\| = \sup_{v \in V, v \neq 0} \frac{\|Lv\|_W}{\|v\|_V}.$$

Remarks.

(1) There are other equivalent definitions of the operator norm $\|L\|$:

$$\begin{aligned} \|L\| &= \sup_{v \in V, \|v\|_V = 1} \|Lv\|_W \\ \|L\| &= \sup_{v \in V, \|v\|_V \leq 1} \|Lv\|_W \\ \|L\| &= \min\{C : (\forall\, v \in V)\ \|Lv\|_W \leq C\|v\|_V\} \end{aligned}$$

(i.e. $\|L\|$ is the smallest "stretching factor upper bound" C).

Exercise. Show these are equivalent to the definition above.

(2) The most common use of the operator norm is the obvious but powerful inequality: $(\forall\, v \in V)\ \|Lv\|_W \leq \|L\| \cdot \|v\|_V$.

Equivalence of Norms

Lemma. If $(V, \|\cdot\|)$ is a normed linear space, then $\|\cdot\| : (V, \|\cdot\|) \to \mathbb{R}$ is continuous.

Proof. For $v_1, v_2 \in V$, $\|v_1\| = \|v_1 - v_2 + v_2\| \leq \|v_1 - v_2\| + \|v_2\|$, and thus $\|v_1\| - \|v_2\| \leq \|v_1 - v_2\|$. Similarly, $\|v_2\| - \|v_1\| \leq \|v_2 - v_1\| = \|v_1 - v_2\|$. So $|\|v_1\| - \|v_2\|| \leq \|v_1 - v_2\|$. Given $\epsilon > 0$, let $\delta = \epsilon$, etc.

Definition. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, both on the same vector space V, are called *equivalent norms* on V if $\exists$ constants $C_1, C_2 > 0$ for which $(\forall\, v \in V)\ \frac{1}{C_1}\|v\|_2 \leq \|v\|_1 \leq C_2\|v\|_2$.

Remarks.

(1) Two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on V are equivalent iff the identity map $I : (V, \|\cdot\|_\alpha) \to (V, \|\cdot\|_\beta)$ is bicontinuous ($\|v\|_\beta \leq C_1\|v\|_\alpha \Rightarrow I : (V, \|\cdot\|_\alpha) \to (V, \|\cdot\|_\beta)$ is continuous, and $\|v\|_\alpha \leq C_2\|v\|_\beta \Rightarrow I : (V, \|\cdot\|_\beta) \to (V, \|\cdot\|_\alpha)$ is continuous.)

(2) Equivalence of norms (denoted temporarily by $\sim$) is an equivalence relation on the set of all norms on a fixed vector space V: (i) $\|\cdot\|_\alpha \sim \|\cdot\|_\alpha$; (ii) $\|\cdot\|_\alpha \sim \|\cdot\|_\beta$ iff $\|\cdot\|_\beta \sim \|\cdot\|_\alpha$; and (iii) if $\|\cdot\|_\alpha \sim \|\cdot\|_\beta$ and $\|\cdot\|_\beta \sim \|\cdot\|_\gamma$, then $\|\cdot\|_\alpha \sim \|\cdot\|_\gamma$.

For *finite dimensional* vector spaces V, all norms are equivalent.

The Norm Equivalence Theorem

If V is a finite dimensional vector space, then any two norms on V are equivalent.

Proof. Fix a basis $\{v_1, \ldots, v_n\}$ for V, and identify V with $\mathbb{F}^n$ ($v \in V \leftrightarrow x \in \mathbb{F}^n$ where $v = x_1 v_1 + \cdots + x_n v_n$). Using this identification, we can restrict our attention to $\mathbb{F}^n$. Let $|x| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$ denote the euclidean norm [i.e., ℓ^2 norm] on $\mathbb{F}^n$. Because equivalence of norms is an equivalence relation, it suffices to show that any given norm $\|\cdot\|$ on $\mathbb{F}^n$ is equivalent to the euclidean norm $|\cdot|$. For $x \in \mathbb{F}^n$, $\|x\| = \|\sum_{i=1}^n x_i e_i\| \leq \sum_{i=1}^n |x_i| \cdot \|e_i\| \leq (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}} (\sum_{i=1}^n \|e_i\|^2)^{\frac{1}{2}}$ by the Schwarz inequality in $\mathbb{R}^n$. Thus $\|x\| \leq M|x|$, where $M = (\sum_{i=1}^n \|e_i\|^2)^{\frac{1}{2}}$. Thus the identity map $I : (\mathbb{F}^n, |\cdot|) \to (\mathbb{F}^n, \|\cdot\|)$ is continuous, which is half of what we have to show. Composing the map with $\|\cdot\| : (\mathbb{F}^n, \|\cdot\|) \to \mathbb{R}$ (which is continuous by the preceding Lemma), we conclude that $\|\cdot\| : (\mathbb{F}^n, |\cdot|) \to \mathbb{R}$ is continuous. Let $S = \{x \in \mathbb{F}^n : |x| = 1\}$. Then S is compact in $(\mathbb{F}^n, |\cdot|)$, and thus $\|\cdot\|$ takes on its minimum on S, which must be > 0 since $0 \notin S$. Let $m = \min_{\{w:|w|=1\}} \|x\| > 0$. Hence if $|x| = 1$, then $\|x\| \geq m$. For any $x \in \mathbb{F}^n$ with $x \neq 0$, $\left|\frac{x}{|x|}\right| = 1$, so $\left\|\frac{x}{|x|}\right\| \geq m$, i.e. $|x| \leq \frac{1}{m}\|x\|$. So $\|\cdot\|$ and $|\cdot|$ are equivalent.

Remarks.

(1) All norms on a fixed finite dimensional vector space are equivalent. Be careful, though, when studying problems (e.g. in numerical PDE) where there is a sequence of finite dimensional spaces of increasing dimensions: the constants C_1 and C_2 in the equivalence can depend on the dimension (e.g. $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$ in $\mathbb{F}^n$).

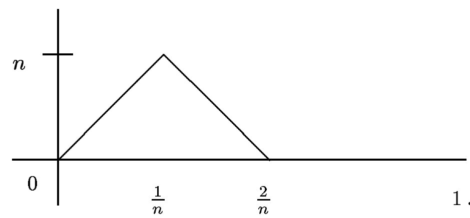
(2) The Norm Equivalence Theorem is *not* true in infinite dimensional vector spaces.

(3) It can be shown that, for a normed linear space V, the closed unit ball $\{v \in V : \|v\| \leq 1\}$ is compact iff $\dim V < \infty$.

Examples.

(1) On $\mathbb{F}_0^\infty = \{x \in \mathbb{F}^\infty : (\exists\, N)(\forall\, n \geq N)\ x_n = 0\}$, for $1 \leq p < q \leq \infty$, the ℓ^p norm and ℓ^q norm are *not* equivalent. We show the case $p = 1$, $q = \infty$. First note that $\|x\|_\infty \leq \sum_{i=1}^\infty |x_i| = \|x\|_1$, so $I : (\mathbb{F}_0^\infty, \|\cdot\|_1) \to (\mathbb{F}_0^\infty, \|\cdot\|_\infty)$ *is* continuous. But if $y_1 = (1, 0, 0 \cdots)$, $y_2 = (1, 1, 0, \cdots)$, $y_3 = (1, 1, 1, 0, \cdots)$, etc., then $\|y_n\|_\infty = 1\, \forall\, n$, but $\|y_n\|_1 = n$. So there does *not* exist a constant C for which $(\forall\, x \in \mathbb{F}_0^\infty)\ \|x\|_1 \leq C\|x\|_\infty$.

(2) On $C([a, b])$, for $1 \leq p < q \leq \infty$, the L^p and L^q norms are *not* equivalent. We will show the case $p = 1$, $q = \infty$ here: $\|u\|_1 = \int_a^b |u(x)|dx \leq \int_a^b \|u\|_\infty dx = (b - a)\|u\|_\infty$, so $I : (C([a, b]), \|\cdot\|_\infty) \to (C([a, b]), \|\cdot\|_1)$ *is* continuous. (Remark: Since the integral $\mathcal{I}(u) = \int_a^b u(x)dx$ is clearly continuous on $(C([a, b]), \|\cdot\|_1)$ since $|\mathcal{I}(u_1) - \mathcal{I}(u_2)| \leq \int_a^b |u_1(x) - u_2(x)|dx = \|u_1 - u_2\|_1$, composition of these two continuous operators implies the standard result that if $u_n \to u$ uniformly on $[a, b]$, then $\int_a^b u_n(x)dx \to \int_a^b u(x)dx$.) WLOG assume $a = 0$, $b = 1$. Let u_n be



Then $\|u_n\|_1 = 1$, but $\|u_n\|_\infty = n$. So there does not exist a constant C for which $(\forall\, u \in C([a,b]))\ \|u\|_\infty \leq C\|u\|_1$.

(3) In ℓ^2 (subspace of $\mathbb{F}^\infty$) with norm $\|x\|_{\ell^2} = \sqrt{\sum_{i=1}^\infty |x_i|^2}$, the closed unit ball $\{x \in \ell^2 : \|x\|_{\ell^2} \leq 1\}$ is *not* compact. The sequence $e_1, e_2, e_3, \ldots$ is bounded $\|e_i\| \leq 1$, and all are in the closed unit ball, but no subsequence converges because $\|e_i - e_j\|_{\ell^2} = \sqrt{2}$ for $i \neq j$.

Exercise. Does the sequence $e_1, e_2, e_3, \ldots$ converge weakly in ℓ^2? (A sequence $\{x_n\}$ is said to converge weakly to x in ℓ^2 if $(\forall\, y \in \ell^2)\ \langle x_n, y\rangle \to \langle x, y\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product $\langle x, y\rangle = \sum_{i=1}^\infty x_i \overline{y_i}$ on ℓ^2.)

Norms induced by inner products

Let V be a vector space and $\langle\cdot,\cdot\rangle$ be an inner product on V. Define $\|v\| = \sqrt{\langle v, v\rangle}$. By the properties of an inner product, $\|v\| \geq 0$ with $\|v\| = 0$ iff $v = 0$, and $(\forall\, \alpha \in \mathbb{F})(\forall\, v \in V)\ \|\alpha v\| = |\alpha| \cdot \|v\|$. To show that $\|\cdot\|$ is actually a norm on V we need the triangle inequality. We begin by first showing the *Cauchy-Schwarz inequality.*

Lemma.[**The Cauchy-Schwarz inequality**] For all $v, w \in V$ we have $|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$. Moreover, we have equality iff v and w are linearly dependent. (This latter statement is sometimes called the "converse of Cauchy-Schwarz.")

Proof.

Case (i) If $v = 0$ or $w = 0$, clear.

Case (ii) If $\|v\| = \|w\| = 1$ and $\langle v, w\rangle \geq 0$, then $0 \leq \|v - w\|^2 = \langle v - w, v - w\rangle = \langle v, v\rangle - 2\mathcal{R}e\langle v, w\rangle + \langle w, w\rangle = 2(1 - \langle v, w\rangle)$ so $\langle v, w\rangle \leq 1$ (with equality iff $v = w$).

Case (iii) For any $v \neq 0$ and $w \neq 0$, choose $\alpha \in \mathbb{F}$ with $|\alpha| = 1$ and $\alpha\langle v, w\rangle \geq 0$. Let $v_1 = \frac{\alpha}{\|v\|}v$ and $w_1 = \frac{w}{\|w\|}$. Then $\|v_1\| = \|w_1\| = 1$ and $\langle v_1, w_1\rangle \geq 0$, so $\frac{|\langle v,w\rangle|}{\|v\|\cdot\|w\|} = \frac{\alpha\langle v,w\rangle}{\|v\|\cdot\|w\|} = \langle v_1, w_1\rangle \leq 1$ (with equality iff $v_1 = w_1$).

Exercise. In case (iii) of the above proof, show v, w are linearly dependent iff $v_1 = w_1$.

Now the triangle inequality follows

$$\begin{aligned}\|v+w\|^2 &= \langle v+w, v+w\rangle = \langle v,v\rangle + 2\mathcal{R}e\langle v,w\rangle + \langle w,w\rangle \\ &\le \|v\|^2 + 2|\langle v,w\rangle| + \|w\|^2 \le \|v\|^2 + 2\|v\|\cdot\|w\| + \|w\|^2 = (\|v\|+\|w\|)^2.\end{aligned}$$

So $\|v\| = \sqrt{\langle v,v\rangle}$ is a norm on V. It is called the norm induced by the inner product $\langle\cdot,\cdot\rangle$. An inner product induces a norm, which induces a metric $(V,\langle\cdot,\cdot\rangle) \leftrightarrow (V,\|\cdot\|) \leftrightarrow (V,d)$.

Examples.

(1) The Euclidean norm [i.e. ℓ^2 norm] on $\mathbb{F}^n$ is induced by the standard inner product $\langle x,y\rangle = \sum_{i=1}^n x_i\overline{y_i}$: $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i\overline{x_i}} = \sqrt{\sum_{i=1}^n |x_i|^2}$.

(2) Let $A \in \mathbb{F}^{n\times n}$ by Hermitian symmetric and positive definite, and let

$$\langle x,y\rangle_A = \sum_{i=1}^n\sum_{j=1}^n x_i a_{ij}\overline{y_j} \quad \text{for } x,y\in\mathbb{F}^n.$$

Then $\langle\cdot,\cdot\rangle_A$ is an inner product on $\mathbb{F}^n$, which induces the norm

$$\|x\|_A = \sqrt{\langle x,x\rangle_A} = \sqrt{\sum_{i=1}^n\sum_{j=1}^n x_i a_{ij}\overline{x_j}} = \sqrt{x^T A\overline{x}} = \sqrt{x^H\overline{A}x}.$$

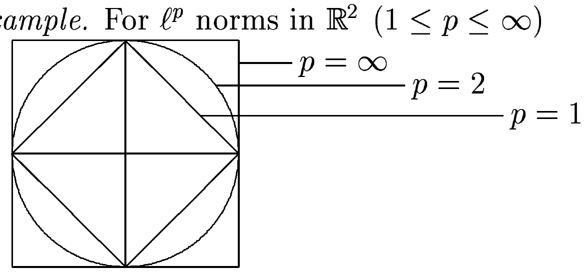
Remark. An alternate convention is to define $\langle x,y\rangle_A$ to be $\sum_{i=1}^n\sum_{j=1}^n \overline{y_i}a_{ij}x_j = y^H Ax$, in which case $\|x\|_A = \sqrt{x^H Ax}$.

(3) The ℓ^2 norm on ℓ^2 (subspace of $\mathbb{F}^\infty$) is induced by the inner product $\langle x,y\rangle = \sum_{i=1}^\infty x_i\overline{y_i}$: $\|x\|_2 = \sqrt{\sum_{i=1}^\infty x_i\overline{x_i}} = \sqrt{\sum_{i=1}^\infty |x_i|^2}$.

(4) The L^2 norm $\|u\|_2 = \left(\int_a^b |u(x)|^2 dx\right)^{\frac{1}{2}}$ on $C([a,b])$ is induced by the inner product $\langle u,v\rangle = \int_a^b u(x)\overline{v(x)}dx$.

Closed unit balls $\{v\in V : \|v\|\le 1\}$ in finite dimensional normed linear spaces V

Example. For ℓ^p norms in $\mathbb{R}^2$ $(1\le p\le\infty)$

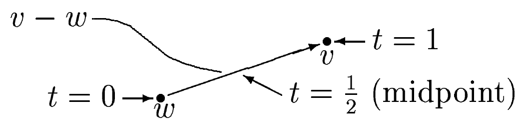


Definition. A subset C of a vector space V is called *convex* if

$$(\forall\, v,w\in C)(\forall\, t\in[0,1]) \qquad tv+(1-t)u\in C.$$

Remarks.

(1) This means that the line segment joining v and w is in C if v and w are in C:



$w + t(v - w)$ is on this line segment.

(2) The linear combination $tv + (1 - t)w$ for $t \in [0, 1]$ is often called a *convex combination* of v and w.

Let $B = \{v \in V : \|v\| \leq 1\}$ denote the closed unit ball in a *finite dimensional* normed linear space.

Facts.

(1) B is convex.

(2) B is compact.

(3) B is symmetric (if $v \in B$ and $\alpha \in \mathbb{F}$ with $|\alpha| = 1$, then $\alpha v \in B$).

(4) The origin is in the interior of B.

Lemma. If $\dim V < \infty$ and $B \subset V$ satisfies the four conditions in the statement of *facts* above, then there is a unique norm on V for which B is the closed unit ball:

$$\|v\| = \inf\{c > 0 : \frac{v}{c} \in B\}.$$

Remark. The condition that 0 be in the interior of a set is independent of the norm: by the norm equivalence theorem, all norms induce the same topology on V, i.e. have the same collection of open sets.

Exercise. Show that the object defined in the lemma above does indeed define a norm, and that B is its closed unit ball. The uniqueness of this norm follows from the fact that in any normed linear space, $\|v\| = \inf\{c > 0 : \frac{v}{c} \in B\}$ where B is the closed unit ball $B = \{v : \|v\| \leq 1\}$. Hence there is a one-to-one correspondence between norms on a finite dimensional vector space and subsets B satisfying the four conditions stated above.

Completeness

Completeness in a normed linear space $(V, \|\cdot\|)$ means completeness in the metric space (V, d), where $d(v, w) = \|v - w\|$: every Cauchy sequence $\{v_n\}$ in V (i.e. $(\forall\, \epsilon > 0)(\exists\, N)(\forall\, n, m \geq N)\ \|v_n - v_m\| < \epsilon$) has a limit in V (i.e. $(\exists\, v \in V)\|v_n - v\| \to 0$ as $n \to \infty$).

Example. $\mathbb{F}^n$ endowed with the euclidean norm $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ is complete.

Topological properties are those which depend only on the collection of open sets (e.g., open, closed, compact, whether a sequence converges, etc.). Completeness is *not* a topological property.

Example. Let $f : [1, \infty) \to (0, 1]$ be given by $f(x) = \frac{1}{x}$ (with the usual metric on $\mathbb{R}$). Then f is a homeomorphism (bijective, bicontinuous), but $[1, \infty)$ *is* complete while $(0, 1]$ is *not* complete.

Completeness is a *uniform property.*

Theorem. *If (X, ρ) and (Y, σ) are metric spaces, and $\varphi(X, \rho) \to (Y, \sigma)$ is a uniform homeomorphism (i.e., bijective, bicontinuous and φ and φ^{-1} are both uniformly continuous), then (X, ρ) is complete iff (Y, σ) is complete.*

The key step in the proof of this theorem is to show that if $\varphi : X \to Y$ is a uniform homeomorphism, then φ preserves Cauchy sequences, i.e. a sequence $\{x_n\}$ is Cauchy in (X, ρ) iff $\{\varphi(X_n)\}$ is Cauchy in (Y, σ). Since bounded linear operators between normed linear spaces are automatically uniformly continuous, several facts follow immediately.

Corollary. If two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are equivalent, then $(V, \|\cdot\|_1)$ is complete iff $(V, \|\cdot\|_2)$ is complete.

Corollary. Every finite dimensional normed linear space is complete.

Proof. If $\dim V = n < \infty$, choose a basis of V and use it to identify V with $\mathbb{F}^n$. Since $\mathbb{F}^n$ is complete in the euclidean norm, the corollary follows from the norm equivalence theorem.

But not every infinite dimensional normed linear space is complete.

Definition. A complete normed linear space is called a *Banach space.* An inner product space for which the induced norm is complete is called a *Hilbert space.*

Examples. To show that a normed linear space is complete, we must show that every Cauchy sequence converges in that space. The basic strategy for showing that a space is complete is a three step process that can be described as follows: given a Cauchy sequence,

(i) construct what you think is its limit;

(ii) show the limit is in the space V;

(iii) show the sequence converges to the limit in V.

(1) Let M be a metric space. Let $C(M)$ denote the vector space of continuous functions $u : M \to \mathbb{F}$. Let $C_b(M)$ denote the subspace of $C(M)$ consisting of all bounded continuous functions $C_b(M) = \{u \in C(M) : (\exists\, K)(\forall\, x \in M)|u(x)| \leq K\}$. On $C_b(M)$, define the sup-norm $\|u\| = \sup_{x \in M} |u(x)|$.

Fact. $(C_b(M)), \|\cdot\|)$ is complete.

Proof. Let $\{u_n\} \subset C_b(M)$ be Cauchy in $\|\cdot\|$. Given $\epsilon > 0$, $\exists\, N$ so that $(\forall\, n, m \geq N)$ $\|u_n - u_m\| < \epsilon$. For each $x \in M$, $|u_n(x) - u_m(x)| \leq \|u_n - u_m\|$, so for each $x \in M$, $\{u_n(x)\}$ is a Cauchy sequence in $\mathbb{F}$, which has a limit in $\mathbb{F}$ (which we will call $u(x)$) since $\mathbb{F}$ is complete: $u(x) = \lim_{n\to\infty} u_n(x)$. Given $\epsilon > 0$, $(\exists\, N)(\forall\, n, m \geq N)(\forall\, x \in M)$ $|u_n(x) - u_m(x)| < \epsilon$. Taking the limit (for each fixed x) as $m \to \infty$, we get $(\forall\, n \geq N)(\forall\, x \in M)$ $|u_n(x) - u(x)| \leq \epsilon$. Thus $u_n \to u$ uniformly, so u is continuous (since the uniform limit of continuous functions is continuous). Clearly u is bounded (choose N for $\epsilon = 1$; then $(\forall\, x \in M)$ $|u(x)| \leq \|u_N\| + 1$), so $u \in C_b(M)$. And now we have $\|u_n - u\| \to 0$ as $n \to \infty$, i.e., $u_n \to u$ in $(C_b(M), \|\cdot\|)$. □

(2) ℓ^p is complete for $1 \leq p \leq \infty$.

$p = \infty$. This is a special case of (1) where $M = \mathbb{N} = \{1, 2, 3, \ldots\}$.

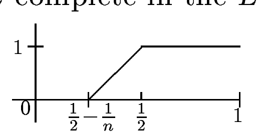
$1 \leq p < \infty$. Let $\{x_k\}$ be a Cauchy sequence in ℓ^p; write $x_k = (x_{k1}, x_{k2}, \ldots)$. Given $\epsilon > 0$, $(\exists\, K)(\forall\, k, \ell \geq K)$ $\|x_k - x_\ell\|_p < \epsilon$. For each $m \in \mathbb{N}$,

$$|x_{km} - x_{\ell m}| \leq \left(\sum_{i=1}^{\infty} |x_{ki} - x_{\ell i}|^p \right)^{\frac{1}{p}} = \|x_k - x_\ell\|,$$

so for each $m \in \mathbb{N}$, $\{x_{km}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$, which has a limit: let $a_m = \lim_{k\to\infty} x_{km}$. Let x be the sequence $x = (a_1, a_2, a_3, \ldots)$; so far, we just know that $x \in \mathbb{F}^\infty$. Given $\epsilon > 0$, $(\exists\, K)(\forall\, k, \ell \geq K)$ $\|x_k - x_\ell\| < \epsilon$. Then for any N and for $k, \ell \geq K$, $\left(\sum_{k=1}^{N} |x_{ki} - x_{\ell i}|^p\right)^{\frac{1}{p}} < \epsilon$; taking the limit as $\ell \to \infty$, $\left(\sum_{i=1}^{N} |x_{ki} - a_i|^p\right)^{\frac{1}{p}} \leq \epsilon$; then taking the limit as $N \to \infty$, $(\sum_{i=1}^{\infty} |x_{ki} - a_i|^p)^{\frac{1}{p}} \leq \epsilon$. Thus $x_K - x \in \ell^p$, so also $x = x_K - (x_K - x) \in \ell^p$, and we have $(\forall\, k \geq K)$ $\|x_k - x\|_p \leq \epsilon$. Thus $\|x_k - x\|_p \to 0$ as $k \to \infty$, i.e., $x_k \to x$ in ℓ^p. □

(3) If M is a compact metric space, then every continuous function $u : M \to \mathbb{F}$ is bounded, so $C(M) = C_b(M)$. In particular, $C(M)$ is complete in the sup norm $\|u\| = \sup_{x\in M} |u(x)|$ (special case of (1).) For example, $C([a, b])$ is complete in the L^∞ norm.

(4) For $1 \leq p < \infty$, $C([a, b])$ is *not* complete in the L^p norm.

Example. On $[0, 1]$, let u_n be: 

Then $u_n \in C[0, 1]$.

Exercise: Show that $\{u_n\}$ is Cauchy in $\|\cdot\|_p$. We must show that there does *not* exist a $u \in C[0, 1]$ for which $\|u_n - u\|_p \to 0$.

Exercise: Show that if $u \in C[0, 1]$ and $\|u_n - u\|_p \to 0$, then $u(x) \equiv 0$ for $0 \leq x < \frac{1}{2}$ and $u(x) \equiv 1$ for $\frac{1}{2} < x \leq 1$, contradicting the continuity of u at $x = \frac{1}{2}$.

(5) $\mathbb{F}_0^\infty = \{x \in \mathbb{F}^\infty : (\exists\, N)(\forall\, n \geq N)\ x_n = 0\}$ is *not* complete in any ℓ^p norm $(1 \leq p \leq \infty)$. This can be shown using the sequences described below.

$\underline{1 \leq p < \infty}$. Choose any $x \in \ell^p \backslash \mathbb{F}_0^\infty$, and consider the truncated sequences $y_1 = (x_1, 0, \ldots)$; $y_2 = (x_1, x_2, 0, \ldots)$; $y_3 = (x_1, x_2, x_3, 0, \ldots)$; etc.

Exercise: Show that $\{y_n\}$ is Cauchy in $(\mathbb{F}_0^\infty, \|\cdot\|_p)$, but that there is no $y \in \mathbb{F}_0^\infty$ for which $\|y_n - y\|_p \to 0$.

$\underline{p = \infty}$. Same idea: choose any $x \in \ell^\infty \backslash \mathbb{F}_0^\infty$ for which $\lim_{i\to\infty} x_i = 0$, and consider the sequence of truncated sequences.

Completion of a Metric Space

Fact. Let (X, ρ) be a metric space. Then there exists a complete metric space $(\bar{X}, \bar{\rho})$ and an "inclusion map" $i : X \to \bar{X}$ for which i is injective, i is an isometry from X to $i[X]$ (i.e. $(\forall\, x, y \in X)\ \rho(x, y) = \bar{\rho}(i(x), i(y)))$, and $i[X]$ is dense in $\bar{X}$. Moreover, all such $(\bar{X}, \bar{\rho})$ are isometrically isomorphic. The metric space $(\bar{X}, \bar{\rho})$ is called the *completion* of (X, ρ).

One way to construct such an $\bar{X}$ is to take equivalence classes of Cauchy sequences in X to be elements of $\bar{X}$.

Representations of Completions

In some situations, the completion of a metric space can be identified with a larger vector space which actually includes X, and whose elements are objects of a similar nature to the elements of X. One example is $\mathbb{R}$ = completion of the rationals $\mathbb{Q}$. The completion of $C([a, b])$ in the L^p norm (for $1 \leq p < \infty$) can be represented as $L^p([a, b])$, the vector space of [equivalence classes of] Lebesgue measurable functions $u : [a, b] \to \mathbb{F}$ for which $\int_a^b |u(x)|^p dx < \infty$, with norm $\|u\|_p = \left(\int_a^b |u(x)|^p dx\right)^{\frac{1}{p}}$.

Fact. A subset of a complete metric space is complete iff it is closed.

Proposition. Let V be a Banach space, and $W \subset V$ be a subspace. The norm on V restricts to a norm on W. We have:

$$W \text{ is complete} \quad \text{iff} \quad W \text{ is closed.}$$

Examples.

(1) $C_0(\mathbb{R}^n) = \{u \in C_b(\mathbb{R}^n) : \lim_{|x|\to\infty} u(x) = 0\}$.

(2) $C_c(\mathbb{R}^n) = \{u \in C_b(\mathbb{R}^n) : (\exists\, K > 0) \ni (\forall\, x \text{ with } |x| \geq K) u(x) = 0\}$.

Remarks.

(1) If M is a metric space and $u : M \to \mathbb{F}$ is a function, define the *support* of u to be the *closure* of $\{x \in M : u(x) \neq 0\}$. The support of a function is automatically closed. The complement of the support of a function is the interior of $\{x \in M : u(x) = 0\}$.

(2) Elements of $C_c(\mathbb{R}^n)$ are continuous functions with *compact support.*

(3) $C_0(\mathbb{R}^n)$ is complete in the sup norm (exercise). This can either be shown directly, or by showing that $C_0(\mathbb{R}^n)$ is a closed subspace of $C_b(\mathbb{R}^n)$.

(4) $C_c(\mathbb{R}^n)$ is *not* complete. In fact, $C_c(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$. So $C_0(\mathbb{R}^n)$ is a representation of the completion of $C_c(\mathbb{R}^n)$ in the sup norm.

Series in normed linear spaces

Let $(V, \|\cdot\|)$ be a normed linear space. Consider a series $\sum_{n=1}^{\infty} v_n$ in V.

Definition. We say the series *converges in* V if $\exists\, v \in V \ni \lim_{N\to\infty} \|S_N - v\| = 0$, where $S_N = \sum_{n=1}^{N} v_n$ is the N^{th} partial sum. We say this series *converges absolutely* if $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Caution: Strictly speaking, if a series "converges absolutely' in a normed linear space, it does not have to converge in that space.

Example. The series $(1, 0 \cdots) + \left(0, \frac{1}{2}, 0 \cdots\right) + \left(0, 0, \frac{1}{4}, 0 \cdots\right)$ "converges absolutely" in $\mathbb{F}_0^\infty$, but it doesn't converge in $\mathbb{F}_0^\infty$.

Proposition. A normed linear space $(V, \|\cdot\|)$ is complete iff every absolutely convergent series actually converges in $(V, \|\cdot\|)$.

Proof Sketch ($\Rightarrow$) Given an absolutely convergent series, show that the sequence of partial sums is Cauchy: for $m > n$ $\|S_m - S_n\| \leq \sum_{j=n+1}^{m} \|v_j\|$.

($\Leftarrow$) Given a Cauchy sequence $\{x_n\}$, choose $n_1, n_2 < \cdots$ inductively so that for $k = 1, 2, \ldots$, $(\forall\, n, m \geq n_k)$ $\|x_n - x_m\| \leq 2^{-k}$. Then in particular $\|x_{n_k} - x_{n_{k+1}}\| \leq 2^{-k}$. Show that the series $x_{n_1} + \sum_{k=2}^{\infty}(x_{n_k} - x_{n_{k-1}})$ is absolutely convergent. Let x be its limit. Show that $x_n \to x$.