Assignment 1. Due Friday, Oct. 3.

Reading: Read through p. 12 in Course Notes.
Horn and Johnson, Chapter 0.
Review linear algebra from Halmos, Nering, or your choice of similar linear algebra text.

1. Let W_1 and W_2 be finite dimensional subspaces of a vector space V and define

$$W_1 + W_2 = \{w_1 + w_2 : \; w_1 \in W_1, \; w_2 \in W_2\}.$$

(a) Show that

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2).$$

(b) Show that the following conditions are equivalent:

(i) $W_1 \cap W_2 = \{0\}$.

(ii) Each vector $v \in W_1 + W_2$ can be written uniquely as $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.

(iii) If $w_1 \in W_1$ and $w_2 \in W_2$ and $w_1 + w_2 = 0$, then $w_1 = w_2 = 0$.

2. Let $V = \{u \in C^4(\mathbf{R}) : \; u'''' + u'' = 0\}$ and let $W \subset V$ be the subspace of functions satisfying in addition $u(\pi) = 0$.

(a) Find bases for V and W.

(b) Show that the map $u \to u'$ defines a linear transformation from V to V. Hence we can restrict the domain to W to obtain a linear transformation $L : W \to V$.

(c) Find the matrix of L with respect to your bases in part (a).

3. Show that if $0 < p < q \leq \infty$, then $\ell^p \subset \ell^q$ but $\ell^p \neq \ell^q$.

4. Let $\mathcal{P}_n$ denote the vector space of polynomials of degree less than or equal to n. Let $x_0, x_1, \ldots x_n$ be $n+1$ distinct points in $\mathbf{R}$ and define a linear functional $f_k \in \mathcal{P}_n^*$ by

$$f_k(p) = p(x_k),$$

for $p \in \mathcal{P}_n$. Let $\{e_0, e_1, \ldots, e_n\}$ be the basis for $\mathcal{P}_n^*$ dual to the basis $\{1, x, \ldots, x^n\}$ for $\mathcal{P}_n$.

(a) Express f_k as a linear combination of $e_0, \ldots, e_n$.

(b) Show that $\{f_0, \ldots, f_n\}$ is a basis for $\mathcal{P}_n^*$. (Hint: Vandermonde matrix)

(c) Write down the basis of $\mathcal{P}_n$ to which $\{f_0, \ldots, f_n\}$ is dual.

5. Let $[a, b] \subset \mathbf{R}$ be a closed bounded interval, and let $\Omega_n = \{x_0, \ldots, x_n\}$ be a fixed partition of $[a, b]$; i.e., $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. To approximate the solution to boundary value problems in differential equations, one often uses *piecewise polynomials*: functions that are equal to polynomials of a specified degree in each subinterval $[x_i, x_{i+1}]$ and usually have some continuity properties throughout the entire interval $[a, b]$.

   (a) Show that the set of functions

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}, \qquad i = 1, \ldots, n-1,$$

   forms a basis for the set of continuous piecewise linear functions φ on $[a, b]$ satisfying $\varphi(a) = \varphi(b) = 0$. [Note that the coordinates of φ with respect to this basis are the values of φ at the interior nodes: $\varphi(x) = \sum_{i=1}^{n-1} \varphi(x_i)\varphi_i(x)$.]

   (b) Write down a basis for the set of C^1 piecewise cubic functions ψ on $[a, b]$ satisfying $\psi(a) = \psi'(a) = \psi(b) = \psi'(b) = 0$. [Hint: Consider two types of basis functions: piecewise cubics ψ_i that are one at node i and zero at all other nodes and whose first derivatives are zero at all of the nodes; and piecewise cubics ω_i that are zero at all of the nodes but have first derivative equal to one at node i and zero at all other nodes. These functions are called *Hermite cubics*.]