Assignment 3. Due Fri., Oct. 17.

Reading: Class Notes, pp. 21–24, 48–52
Chapter 1 in Horn and Johnson.

1. Let $\langle\cdot,\cdot\rangle$ be an inner product on a vector space V over the field $\mathbf{F}$. Prove the polarization identity:

   (i) If $\mathbf{F}=\mathbf{R}$,
   $$\langle x,y\rangle=\frac{1}{4}\left(\langle x+y,x+y\rangle-\langle x-y,x-y\rangle\right).$$

   (ii) If $\mathbf{F}=\mathbf{C}$,
   $$\langle x,y\rangle=\frac{1}{4}\left(\langle x+y,x+y\rangle-\langle x-y,x-y\rangle+\imath\langle x+\imath y,x+\imath y\rangle-\imath\langle x-\imath y,x-\imath y\rangle\right).$$

   [Note: This shows that an inner product is determined by the associated quadratic form $\langle x,x\rangle$.]

2. Let V be a vector space over a field $\mathbf{F}$. Show that if $\mathbf{F}$ is the field of complex numbers and if V is finite dimensional then every linear transformation $L\in\mathcal{L}(V)$ must have an eigenvalue. (Hint: Consider the matrix of L in a basis $\mathcal{B}$ for V.) Give examples to show that if V is finite dimensional but $\mathbf{F}=\mathbf{R}$, or if V is infinite dimensional and $\mathbf{F}=\mathbf{C}$, then L may not have an eigenvalue.

3. An *upper Hessenberg* matrix $H\in\mathbf{C}^{n\times n}$ is one for which $h_{ij}=0$ for $i>j+1$:
   $$H=\begin{pmatrix} * & * & \dots & * \\ * & * & \dots & * \\ & \ddots & \ddots & \vdots \\ & & * & * \end{pmatrix},$$
   where the $*$'s are arbitrary entries and the other entries are 0. The matrix H is called *unreduced* if $h_{j+1,j}\neq 0$ for $1\le j\le n-1$.

   (a) Show that the eigenvalues of an unreduced upper Hessenberg matrix each have geometric multiplicity one. (Hint: Show that for any $\lambda\in\mathbf{C}$, $\lambda I-H$ has rank at least $n-1$.)

   (b) Show that if the unreduced upper Hessenberg matrix H is *Hermitian*, that is, if it is actually a Hermitian *tridiagonal* matrix of the form
   $$\begin{pmatrix} \alpha_1 & \bar\beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \bar\beta_{n-1} \\ & & \beta_{n-1} & \alpha_n \end{pmatrix},$$
   where each $\beta_j\neq 0$, then its eigenvalues must be distinct.

4. A *companion matrix* $C \in \mathbf{C}^{n\times n}$ has the form

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix},$$

where entries not shown are 0.

(a) Show that the characteristic polynomial of C is

$$p_C(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0.$$

(b) Let $H \in \mathbf{C}^{n\times n}$ be an unreduced upper Hessenberg matrix. Show that H is similar to a companion matrix. (Hint: First show that the vectors $\{e_1, He_1, \ldots, H^{n-1}e_1\}$ (where e_1 is the first unit vector) are linearly independent and then consider the action of H on these basis vectors.)

5. The following describes an iterative procedure called the *power method* for (usually) finding the largest eigenvalue (in magnitude) and an associated eigenvector for $A \in \mathbf{C}^{n\times n}$. Although the method applies in a more general setting, assume for simplicity that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A satisfy $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$, and that A is diagonalizable. Given $u^{(0)} \in \mathbf{C}^n$, define sequences $\{v^{(k)}\}$ and $\{u^{(k)}\}$ in $\mathbf{C}^n$ inductively by

$$v^{(k)} = Au^{(k-1)}, \quad u^{(k)} = v^{(k)}/\max(v^{(k)}), \quad k = 1, 2, \ldots,$$

where $\max(v)$ denotes the element of maximum modulus.

Show that for "almost all" $u^{(0)} \in \mathbf{C}^n$, the sequence $\{u^{(k)}\}$ converges to an eigenvector of A corresponding to λ_1 and $\max(v_k) \to \lambda_1$ as $k \to \infty$; that is, explicitly state a condition (namely that $u^{(0)}$ be outside a certain hyperplane in $\mathbf{C}^n$) that will ensure convergence to the desired eigenpair.