Assignment 4. Due Monday, Oct. 27.

Reading: Class Notes, pp. 52–60
Chapter 2 in Horn and Johnson.

1. **(a)** Let $A \in \mathbf{C}^{n\times n}$ be normal and suppose all eigenvalues of A are real. Show that A is Hermitian.

   (b) Let $A \in \mathbf{C}^{n\times n}$ be normal and suppose all eigenvalues of A have absolute value 1. Show that A is unitary.

   (c) Show by examples that (a) and (b) both fail if the normality assumption is replaced by the weaker assumption that A is diagonalizable.

2. Use the Schur Triangularization Theorem to show that every matrix $A \in \mathbf{C}^{n\times n}$ is "almost" diagonalizable in the following two senses:

   (a) Given $\epsilon > 0$, there is a matrix $\tilde{A} \in \mathbf{C}^{n\times n}$ with distinct eigenvalues for which $\|A - \tilde{A}\|_F < \epsilon$.

   (b) Given $\epsilon > 0$, there is an upper triangular matrix T similar to A for which $|t_{ij}| < \epsilon$ for all $i < j$.

3. **(a)** What are the possible Jordan canonical forms of A if $p_A(t) = (t+4)^3(t-2)^2$?

   (b) Find the Jordan canonical form for

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

4. Let $A, B \in \mathbf{C}^{n\times n}$ be diagonalizable. Show that A and B are *simultaneously diagonalizable* (that is, there exists one invertible matrix S for which both $S^{-1}AS$ and $S^{-1}BS$ are diagonal) if and only if $AB = BA$. Proceed as follows:

   (a) Show that if A and B are simultaneously diagonalizable then $AB = BA$.

   (b) Suppose $AB = BA$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A, with eigenspaces $E_1, \ldots, E_k$ and associated projections $P_1, \ldots, P_k$. Show that $BE_i \subset E_i$ for each i, and deduce that $BP_i = P_iB$ for each i.

   (c) Suppose $AB = BA$. Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbf{C}^n$ consisting of eigenvectors of B. Show that for each i, $1 \le i \le k$, the vectors $\{P_iv_1, \ldots, P_iv_n\}$ span E_i, where E_i and P_i are as in part (b). Also show that each nonzero vector P_iv_j is an eigenvector of B.

(d) Suppose $AB = BA$. Deduce that there is a basis of $\mathbf{C}^n$ consisting of vectors which are eigenvectors of both A and B. Conclude that A and B are simultaneously diagonalizable.

5. Do Problem 1 on p. 76 of Horn and Johnson.

6. Prove the real version of the Schur Theorem: If $A \in \mathbf{R}^{n \times n}$, there is an orthogonal matrix $V \in \mathbf{R}^{n \times n}$ such that

$$V^T A V = \begin{pmatrix} A_1 & \dots & * \\ & \ddots & \vdots \\ & & A_k \end{pmatrix},$$

is *block* upper triangular, where each diagonal block A_i is either a real 1×1 matrix or a real 2×2 matrix whose eigenvalues are a complex conjugate pair $\lambda \neq \bar{\lambda}$.