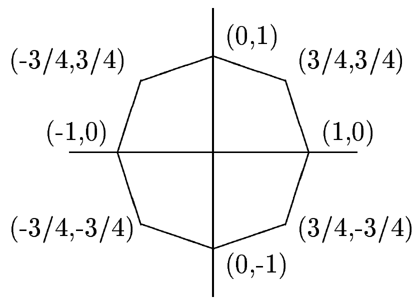
Assignment 7. Due **Fri., Nov. 21**.

Reading: Class Notes, pp. 41–47
Chapter 5 in Horn and Johnson.

1. Pictured below is the unit ball for a certain norm on $\mathbf{R}^2$.



   (a) Determine the unit ball for the dual norm and plot it labelling important boundary points.

   (b) Does the norm whose unit ball is pictured above come from an inner product? Explain why or why not.

2. **(a)** Let $A \in \mathbf{C}^{n\times n}$ be Hermitian and positive definite and define the A-norm of a vector $x \in \mathbf{C}^n$ by $\|x\|_A = (x^H Ax)^{1/2}$. What are the best constants m and M such that

$$m\|x\|_2 \leq \|x\|_A \leq M\|x\|_2, \quad (\forall x \in \mathbf{C}^n)?$$

   (b) Find the best constants m and M such that

$$m|||A|||_1 \leq |||A|||_2 \leq M|||A|||_1, \quad (\forall A \in \mathbf{C}^{n\times n}),$$

   where $|||\cdot|||_p$ denotes the operator norm induced by the ℓ^p norm on $\mathbf{C}^n$: $|||A|||_p = \sup_{\|x\|_p=1} \|Ax\|_p$.

   (c) Find the best constants m and M such that

$$m\|A\|_F \leq |||A|||_2 \leq M\|A\|_F, \quad (\forall A \in \mathbf{C}^{n\times n}),$$

   where $\|\cdot\|_F$ denotes the Frobenius norm: $\|A\|_F = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$.

3. We showed in an earlier homework exercise that the roots of the polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ are the eigenvalues of its *companion matrix*:

$$C(p) = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix},$$

(a) If $|||\cdot|||$ is any submultiplicative norm on $\mathbf{C}^{n\times n}$ and λ is an eigenvalue of $A \in \mathbf{C}^{n\times n}$, show that $|||A||| \geq |\lambda|$.

(b) Take $|||\cdot||| = |||\cdot|||_\infty$ and $A = C(p)$ in part (a) to deduce *Cauchy's bound*: All roots z of $p(z)$ satisfy $|z| \leq 1 + \max\{|a_0|, \ldots, |a_{n-1}|\}$.

(c) Take $|||\cdot||| = |||\cdot|||_1$ and $A = C(p)$ in part (a) to deduce *Montel's bound*: All roots z of $p(z)$ satisfy $|z| \leq \max\{1, |a_0| + |a_1| + \ldots + |a_{n-1}|\}$.

(d) Apply (c) to the polynomial $(z-1)p(z)$ to deduce another bound of Montel: All roots z of $p(z)$ satisfy $|z| \leq |a_0| + |a_1 - a_0| + \ldots + |a_{n-1} - 1|$.

(e) Use (d) to prove *Kakeya's Theorem*: If $f(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ is a polynomial with real nonnegative coefficients satisfying $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 \geq 0$, then all roots of f lie in the closed unit disk in $\mathbf{C}$.

4. **(a)** Do the exercise on p. 46 in the Notes.

(b) Suppose that $A \in \mathbf{C}^{n\times n}$ is invertible, x is the solution of the linear system $Ax = b$, and $\hat{x}$ is the solution of the linear system $A\hat{x} = \hat{b}$. Show that for any matrix norm $\|\cdot\|$ induced by a vector norm (denoted in the same way)

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa(A)\frac{\|b - \hat{b}\|}{\|b\|},$$

where $\kappa(A) = \|A\|\cdot\|A^{-1}\|$ is the condition number of A, and determine conditions under which equality will hold.