Assignment 6. Due Friday, Feb. 27.

Reading: Course Notes, through p. 80 in chapter on Lebesgue integration.
Jones, ch. 2.

1. A collection $A_1, A_2, \ldots$ of measurable subsets of $\mathbf{R}^n$ is said to be *almost disjoint* if $\lambda(A_j \cap A_k) = 0$ for $j \neq k$.

   (a) Prove that if $A_1, A_2, \ldots$ are almost disjoint then $\lambda(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$.

   (b) Conversely, suppose that the measurable sets $A_1, A_2, \ldots$ satisfy $\lambda(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \lambda(A_k) < \infty$. Prove that the sets are almost disjoint. Does this remain true if the $< \infty$ above is replaced by the weaker hypothesis that $\lambda(A_k) < \infty$ for each k?

   (c) Suppose $A_1, A_2, \ldots$ are measurable sets and suppose that each point $x \in \mathbf{R}^n$ belongs to no more than d of the A_k's, where d is a fixed positive integer. Prove that $\sum_{k=1}^{\infty} \lambda(A_k) \leq d\lambda(\cup_{k=1}^{\infty} A_k)$. [If you need a hint, see Jones pp. 58-59.]

2. Define the function f on $(0, 1)$ as follows:

$$f(0.a_1a_2a_3\ldots) = \begin{cases} \frac{1}{n} & \text{if } n \text{ is the smallest integer so that } a_n = 7 \\ 0 & \text{if } a_n \neq 7 \text{ for all } n \end{cases}$$

   [Here $0.a_1a_2a_3\ldots$ is the decimal expansion. In case of nonuniqueness – e.g., $0.5000\ldots = 0.4999\ldots$ – choose the terminating expansion.] Show that f is measurable. What is $\int_0^1 f(x)\,dx$? [The Taylor series for $\ln(1-x)$ will be useful.]

3. **(a)** Show that the function $\sin x/x$ is not integrable on $(0, \infty)$.

   (b) Show that $\lim_{R\to\infty} \int_0^R \sin x/x\,dx$ exists.

   (c) Show that if $f \geq 0$ is measurable on $(0, \infty)$, then $\lim_{R\to\infty} \int_0^R f(x)\,dx < \infty$ if and only if f is integrable on $(0, \infty)$.

4. Evaluate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ by computing $\iint_{\mathbf{R}^2} e^{-(x^2+y^2)}\,dx\,dy$ in two ways:

   (a) by directly using Fubini's theorem, and

   (b) by using polar coordinates in $\mathbf{R}^2$.

5. For the function $f(x, y) = (x-y)/(x+y)^3$, show that the iterated integrals $\int_0^1 \left(\int_0^1 f(x, y)\,dy\right) dx$ and $\int_0^1 \left(\int_0^1 f(x, y)\,dx\right) dy$ both exist but are not equal. Show that this does not violate Fubini's theorem by showing that $\iint_{[0,1]\times[0,1]} |f(x, y)|\,dx\,dy = \infty$.