Assignment 7. Due Friday, Mar. 13.

Reading: Finish Course Notes for this quarter.

1. **(a)** Let $(V, \|\cdot\|)$ be a finite dimensional normed vector space, let $C \subset V$ be a closed nonempty subset, and let $x \in V$. Show that there is a closest point in C to x. Show, however, that the closest point in C need not be unique, even if C is convex.

   (b) Define $\ell_0^\infty = \{(x_1, x_2, \ldots) \in \ell^\infty : \lim_{n\to\infty} x_n = 0\}$. Show that ℓ_0^∞ is a closed subspace of ℓ^∞, so is itself a Banach space. Let $L = \{(x_1, x_2, \ldots) \in \ell_0^\infty : \sum_{n=1}^\infty 2^{-n} x_n = 1\}$. Show that L is a closed hyperplane in ℓ_0^∞. Show, however, that there is no element of L of smallest norm; i.e., there is no closest point in l to the origin.

2. Let $\{p_n(x)\}$ be obtained from $\{x^n : n = 0, 1, \ldots\}$ by Gram-Schmidt orthogonalization in $L^2[-1, 1]$. [$\{\sqrt{\frac{2}{2n+1}} p_n\}$ are called the Legendre polynomials.]

   (a) Show that $\{p_n\}$ is a complete orthonormal set. [Hint: Use the Weierstrass Approximation Theorem.]

   (b) Compute p_0, p_1, and p_2.

   (c) Use (a), (b), and Hilbert space theory to find constants $a, b, c \in \mathbf{R}$ for which $\int_{-1}^1 |x^3 - (a + bx + cx^2)|^2\, dx$ is minimized.

3. Let $\lambda \in \mathbf{C}$. Show that if there is a nonzero solution of $v'' = -\lambda v$ which is 2π-periodic, then $\lambda = n^2$ for some $n \in \{0, 1, 2, \ldots\}$.

4. Apply Parseval's relation to the function $f(x) = x$ on $(-\pi, \pi)$ to evaluate $\sum_{n=1}^\infty \frac{1}{n^2}$.

5. **(a)** Show that $\{\sqrt{\frac{2}{\pi}} \sin(nx) : \quad n = 1, 2, \ldots\}$ is a complete orthonormal system in $L^2(0, \pi)$. The rest of this problem analyzes the Fourier series solution of the vibrating string problem:

$$\text{DE: } u_{tt} = u_{xx}, \quad 0 \le x \le \pi, \quad t \ge 0,$$

$$\text{IC: } u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 \le x \le \pi,$$

$$\text{BC: } u(0, t) = u(\pi, t) = 0, \quad t \ge 0.$$

   (b) Show that if $f \in C^1[0, \pi]$ satisfies $f(0) = f(\pi) = 0$, then the expansion of f in terms of the orthonormal basis in (a) converges uniformly to f on $[0, \pi]$. (This is called the Fourier sine series.)

   (c) Show directly that if $f \in C^2[0, \pi]$, then the series for u obtained by superposing fundamental modes converges uniformly on $[0, \pi] \times \mathbf{R}$ to a continuous function $u(x, t)$ which satisfies $u(x, 0) = f(x)$.

(d) Show that u from part (c) agrees with D'Alembert's solution of this IBVP, and hence show that $u \in C^2([0, \pi] \times \mathbf{R})$ and that u satisfies the wave equation, the IC, and the BC. (Observe that there are difficulties in trying to justify term by term differentiation of the series for u to check that u satisfies the wave equation. Find conditions on f (smoothness, values of f, f', etc. at 0, π) which would justify this approach.)