THE RANGE AND THE NULL SPACE OF A MATRIX

Suppose that A is an $m \times n$ matrix with real entries. There are two important subspaces associated to the matrix A. One is a subspace of $\boldsymbol{R}^m$. The other is a subspace of $\boldsymbol{R}^n$. We will assume throughout that all vectors have real entries.

THE RANGE OF A.

The range of A is a subspace of $\boldsymbol{R}^m$. We will denote this subspace by $\mathcal{R}(A)$. Here is the definition:

$$\mathcal{R}(A) = \{Y : \textit{there exists at least one } X \textit{ in } \boldsymbol{R}^n \textit{ such that } AX = Y \}$$

THEOREM. If A is an $m \times n$ matrix, then $\mathcal{R}(A)$ is a subspace of $\boldsymbol{R}^m$.

Proof. First of all, notice that if Y is in $\mathcal{R}(A)$, then $Y = AX$ for some X in $\boldsymbol{R}^n$. Since A is $m \times n$ and X is $n \times 1$, $Y = AX$ will be $m \times 1$. That is, Y will be in $\boldsymbol{R}^m$. This shows that the set $\mathcal{R}(A)$ is a subset of $\boldsymbol{R}^m$.

Now we verify the three subspace requirements. Let $W = \mathcal{R}(A)$.

(a) Let $\mathbf{0}_m$ denote the zero vector in $\boldsymbol{R}^m$ and $\mathbf{0}_n$ denote the zero vector in $\boldsymbol{R}^n$. Notice that $A\mathbf{0}_n = \mathbf{0}_m$. Hence $AX = \mathbf{0}_m$ is satisfied by at least one X in $\boldsymbol{R}^n$, namely $X = \mathbf{0}_n$. Thus, $\mathbf{0}_m$ is indeed in W and hence requirement (a) is valid for W.

(b) Suppose that Y_1 and Y_2 are in W. This means that each of the matrix equations $AX = Y_1$ and $AX = Y_2$ has at least one solution. Suppose that $X = X_1$ is a vector in $\boldsymbol{R}^n$ satisfying the first equation. That is, $AX_1 = Y_1$. Suppose that $X = X_2$ is a vector in $\boldsymbol{R}^n$ satisfying the second equation. That is, $AX_2 = Y_2$. Now consider the matrix equation $AX = Y_1 + Y_2$. Let $X = X_1 + X_2$, a vector in $\boldsymbol{R}^n$. Then we have

$$AX \ = \ A(X_1 + X_2) \ = \ AX_1 \ + \ AX_2 \ = \ Y_1 + Y_2$$

Therefore, $Y_1 + Y_2$ is in W. This shows that W is closed under addition and so requirement (b) is valid for W.

(c) Suppose that Y_1 is in W. Let c be any scalar. Since Y_1 is in W, there exists a vector X_1 in $\boldsymbol{R}^n$ such that $AX_1 = Y_1$. Now consider the matrix equation $AX = cY_1$. Let $X = cX_1$, a vector in $\boldsymbol{R}^n$. Then we have

$$AX \ = \ A(cX_1) \ = \ c(AX_1) \ = \ cY_1$$

Therefore, cY_1 is in W. Therefore, $Y_1 + Y_2$ is in W. This shows that W is closed under scalar multiplication and so requirement (c) is valid for W.

We have proved that $W = \mathcal{R}(A)$ is a subset of $\boldsymbol{R}^m$ satisfying the three subspace requirements. Hence $\mathcal{R}(A)$ is a subspace of $\boldsymbol{R}^m$.

THE NULL SPACE OF A.

The null space of A is a subspace of $\boldsymbol{R}^n$. We will denote this subspace by $\mathcal{N}(A)$. Here is the definition:

$$\mathcal{N}(A) = \{X : AX = \mathbf{0}_m \}$$

THEOREM. If A is an $m \times n$ matrix, then $\mathcal{N}(A)$ is a subspace of $\boldsymbol{R}^n$.

Proof. First of all, notice that if X is in $\mathcal{N}(A)$, then $AX = \mathbf{0}_m$. Since A is $m \times n$ and AX is $m \times 1$, it follows that X must be $n \times 1$. That is, X is in $\boldsymbol{R}^n$. Therefore, $\mathcal{N}(A)$ is a subset of $\boldsymbol{R}^n$.

Now we verify the three subspace requirements. Let $W = \mathcal{N}(A)$.

(a) Notice that $A\mathbf{0}_n = \mathbf{0}_m$. Hence the equation $AX = \mathbf{0}_m$ is satisfied by $X = \mathbf{0}_n$. It follows that $\mathbf{0}_n$ is indeed in W.

(b) Suppose that X_1 and X_2 are in W. This means that $AX_1 = \mathbf{0}_m$ and $AX_2 = \mathbf{0}_m$. Let $X = X_1 + X_2$. Then

$$AX \;=\; A(X_1 + X_2) \;=\; AX_1 + AX_2 \;=\; \mathbf{0}_m + \mathbf{0}_m = \mathbf{0}_m$$

Therefore, $X = X_1 + X_2$ is in W. This shows that W is closed under addition and so requirement (b) is valid for W.

(c) Suppose that X_1 is in W. Let c be any scalar. Since X_1 is in W, we have $AX_1 = \mathbf{0}_m$. Let $X = cX_1$. Then

$$AX \;=\; A(cX_1) \;=\; c(AX_1) \;=\; c\mathbf{0}_m \;=\; \mathbf{0}_m$$

Therefore, $X = cX_1$ is in W. This shows that W is closed under scalar multiplication and so requirement (c) is valid for W.

We have proved that $W = \mathcal{N}(A)$ is a subset of $\boldsymbol{R}^n$ satisfying the three subspace requirements. Hence $\mathcal{N}(A)$ is a subspace of $\boldsymbol{R}^n$.

THE DIMENSION THEOREM: If A is an $m \times n$ matrix, then $dim\big(\mathcal{N}(A)\big) + dim\big(\mathcal{R}(A)\big) \;=\; n.$