SECTION 1.1:

PROBLEMS 2, 5, and 6. An equation with two unknowns x_1 and x_2 is a linear equation if it can be written in the form $a_1x_1 + a_2x_2 = b$, where a_1, a_2, and b are constants. Thus, the equation $x_1x_2 + x_2 = 1$ is not linear because of the first term. The equation $|x_1| - |x_2| = 0$ is not linear because $|x_1| = x_1$ or $-x_1$, depending on the sign of x_1, and so $|x_1|$ is not of the form a_1x_1, where a_1 is a constant. However, the equation $\pi x_1 + \sqrt{7}x_2 = \sqrt{3}$ is linear because π, $\sqrt{7}$, and $\sqrt{3}$ are constants.

PROBLEM 19. The specified 2×3 matrix is

$$A = \begin{bmatrix} 2 & 1 & 6 \\ 4 & 3 & 8 \end{bmatrix}$$

PROBLEM 20. The specified 2×4 matrix is

$$C = \begin{bmatrix} 1 & 2 & 7 & 1 \\ 2 & 2 & 4 & 3 \end{bmatrix}$$

PROBLEM 26.

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The augmented matrix is

$$B = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 5 & 1 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

SECTION 1.2:

PROBLEM 7. The matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

is in echelon form. We reduce it to its reduced echelon form as follows:

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -10 & -5 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The elementary row operations performed above are: We added (-3) × row 2 to row 1, we added 10 × row 3 to row 1, and we added (-4) × row 3 to row 2. The final matrix

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

is in reduced echelon form.

PROBLEM 17. The given matrix is already in echelon form. We reduce it to reduced echelon form as follows:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We first added (-1) × row 2 to row 1. We then added 1 × row 3 to row 1. Finally, we added (-1) × row 3 to row 2. The last matrix is in reduced echelon form.

PROBLEM 18. The matrix

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not in echelon form, but if we multiply row 2 by 1/2, we obtain the following matrix which is in echelon form:

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we now add (-3) × row 2 to row 1, we obtain the following matrix which is in reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

PROBLEM 27. This problem concerns the system of equations:

$x_1 + x_2 - x_3 = 2$
$-3x_1 - 3x_2 + 3x_3 = -6$

The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ -3 & -3 & 3 & -6 \end{bmatrix}$$

Adding 3 $\times$ row 1 to row 2, we obtain

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in reduced echelon form. This is the augmented matrix for the system of equations

$x_1 + x_2 - x_3 = 2$
$0x_1 + 0x_2 + 0x_3 = 0$

We see that x_1 is the leading variable and that x_2 and x_3 are the free variables. The solutions to the system of equations given in this problem can be described as follows:

$(x_1, x_2, x_3) = (2 - x_2 + x_3, x_2, x_3)$

where x_2 and x_3 are arbitrary.

PROBLEM 28. This problem concerns the system of equations:

$2x_1 + 3x_2 - 4x_3 = 3$
$x_1 - 2x_2 - 2x_3 = -2$
$-x_1 + 16x_2 + 2x_3 = 16$

The corresponding augmented matrix is

$$\begin{bmatrix} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{bmatrix}$$

We will find the reduced echelon form for this matrix:

$$\begin{bmatrix} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & -2 & -2 \\ 2 & 3 & -4 & 3 \\ -1 & 16 & 2 & 16 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & -2 & -2 \\ 0 & 7 & 0 & 7 \\ 0 & 14 & 0 & 14 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 14 & 0 & 14 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We performed the following sequence of elementary row operations: Interchanging row 1 and row 2, adding (-2) $\times$ row 1 to row 2 and adding 1 $\times$ row 1 to row 3, multiplying row 2 by 1/7, adding -14 $\times$ row 2 to row 3, and finally adding 2 $\times$ row 2 to row 1. The last matrix is the augmented matrix for

$$\begin{aligned} 1x_1 + 0x_2 - 2x_3 &= 0 \\ 0x_1 + 1x_2 + 0x_3 &= 1 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

The system of equations in this problem is consistent, x_1 and x_2 are the leading variables, x_3 is the free variable. The solutions can be described as follows:

$$(x_1, x_2, x_3) = (2x_3, 1, x_3)$$

where x_3 is arbitrary.

SECTION 1.3

PROBLEM 1.

the augmented matrix for the given system is:

$$\begin{bmatrix} 2 & 2 & -1 & 1 \\ -2 & -2 & 4 & 1 \\ 2 & 2 & 5 & 5 \\ -2 & -2 & -2 & -3 \end{bmatrix}$$

Adding row 1 to row 2, adding (-1) × row 1 to row 3, and adding row 1 to row 4 gives the matrix

$$\begin{bmatrix} 2 & 2 & -1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

Multiplying row 1 by 1/2 and multiplying row 2 by 1/3 gives:

$$\begin{bmatrix} 1 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

Adding 1/2 × row 2 to row 1, (-6) × row 2 to row 3, and 3 × row 2 to row 4 gives

$$\begin{bmatrix} 1 & 1 & 0 & 5/6 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in reduced echelon form. We have n=3, r=2, and n-r = 1. There is one independent variable (also called a free variable), namely the variable x_2.

PROBLEM 2

The augmented matrix is

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 5 & 4 \\ 4 & 2 & -2 \end{bmatrix}$$

This is row-equivalent to the following matrices:

$$\begin{bmatrix} 1 & 1 & 1/2 \\ 4 & 5 & 4 \\ 4 & 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced echelon form. We have n = 2, r=2, and n-r = 0. There are no independent variables.

PROBLEM 4

The augmented matrix for this system of equations is:

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 5 & 2 \\ 2 & 4 & 6 & 1 & 1 \\ -1 & -2 & -3 & 7 & 2 \end{bmatrix}$$

This matrix is row-equivalent to the following matrices:

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -9 & -3 \\ 0 & 0 & 0 & 9 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & -9 & -3 \\ 0 & 0 & 0 & 9 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced echelon form. We have n=4, r=2, and n-r=2. the independent (i.e., free) variables are x_2 and x_3.

PROBLEM 6. The system of equations is a system of 3 equations in 4 unknowns. The coefficient matrix A is a 3×4 matrix. Then r = rank(A) satisfies the following inequality $0 \leq \text{r} \leq 3$. The possibilities for r are r = 0, 1, 2, or 3. Thus, the possibilities for n-r are n-r = 4, 3, 2, or 1.

A consistent system of linear equations has a unique solution (i.e. exactly one solution) if and only if r = n. Since n=4 and r $\leq$ 3 for the system of equations system considered in this problem, it is not possible to have a unique solution.

PROBLEM 14. The given information implies that the system of equations is consistent. It is a system of 3 equations in 4 unknowns, just as in PROBLEM 6. It cannot have a unique solution. Thus, the system of equations must have infinitely many solutions.

PROBLEM 16. The system of equations is homogeneous, hence consistent. It could have either exactly one solution or infinitely many solutions.

PROBLEM 18. The system of equations is homogeneous. One solution is the trivial solution $x_1 = x_2 = x_3 = 0$. But the system of equations also has a nontrivial solution, namely $x_1 = 1, x_2 = 3, x_3 = -1$. Thus, the system must have infinitely many solutions.

PROBLEM 20. The specified system of equations is homogeneous, hence consistent. But it is a system of 3 equations in 4 unknowns. As explained in problem 6 above, it cannot have a unique solution. Hence this system of equations must have infinitely many solutions. It cannot have just the trivial solution. It must have nontrivial solutions.

SECTION 1.5

PROBLEM 2.

$$(a) \quad B+C = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & 4 \end{bmatrix}$$

$$(b) \quad 3A = 3\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 9 \end{bmatrix}$$

$$(c) \quad A{+}2C = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + 2\begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 3 & 5 \end{bmatrix}$$

$$(d) \quad C+8Z = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}$$

PROBLEM 8.

$$(a) \quad \mathbf{t}+\mathbf{s} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$(b) \quad \mathbf{r}+3\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \end{bmatrix} = \begin{bmatrix} -11 \\ 18 \end{bmatrix}$$

$$(c) \quad 2\mathbf{u}+3\mathbf{t} = 2\begin{bmatrix} -4 \\ 6 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 12 \end{bmatrix} + \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} -5 \\ 24 \end{bmatrix}$$

PROBLEM 20. First, we have

$$3\mathbf{r}+2\mathbf{s} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

Hence, the equation in this problem becomes:

$$a_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad + \quad a_2 \begin{bmatrix} -4 \\ 6 \end{bmatrix} \quad = \quad \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

This is the vector equation which is equivalent to the follow system of equations:

$$\begin{array}{rcrcr} 1a_1 & - & 4a_2 & = & 7 \\ 4a_1 & + & 6a_2 & = & -6 \end{array}$$

This is a system of 2 equations in the 2 unknowns a_1 and a_2. We can solve this system of equations by Gauss-Jordan elimination. The corresponding augmented matrix is

$$\begin{bmatrix} 1 & -4 & 7 \\ 4 & 6 & -6 \end{bmatrix}$$

This is row equivalent to:

$$\begin{bmatrix} 1 & -4 & 7 \\ 0 & 22 & -34 \end{bmatrix}, \quad \begin{bmatrix} 1 & -4 & 7 \\ 0 & 1 & -17/11 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 9/11 \\ 0 & 1 & -17/11 \end{bmatrix}$$

We see that there is a unique solution, namely $a_1 = 9/11, \quad a_2 = -17/11$.

PROBLEM 31.

$$AB \quad = \quad \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \quad = \quad \begin{bmatrix} 2 \times 1 + 3 \times 1 & 2 \times 2 + 3 \times 4 \\ 1 \times 1 + 4 \times 1 & 1 \times 2 + 4 \times 4 \end{bmatrix} \quad = \quad \begin{bmatrix} 5 & 16 \\ 5 & 18 \end{bmatrix}$$

$$BA \quad = \quad \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad = \quad \begin{bmatrix} 1 \times 2 + 2 \times 1 & 1 \times 3 + 2 \times 4 \\ 1 \times 2 + 4 \times 1 & 1 \times 3 + 4 \times 4 \end{bmatrix} \quad = \quad \begin{bmatrix} 4 & 11 \\ 6 & 19 \end{bmatrix}$$

PROBLEM 36

$$B\mathbf{u} \quad = \quad \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad = \quad \begin{bmatrix} 1 \times 1 + 2 \times 3 \\ 1 \times 1 + 4 \times 3 \end{bmatrix} \quad = \quad \begin{bmatrix} 7 \\ 13 \end{bmatrix}$$

PROBLEM 38.

$$CB = \begin{bmatrix} 2 & 1 \\ 4 & 0 \\ 8 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2\times 1 + 1\times 1 & 2\times 2 + 1\times 4 \\ 4\times 1 + 0\times 1 & 4\times 2 + 0\times 4 \\ 8\times 1 + (-1)\times 1 & 8\times 2 + (-1)\times 4 \\ 3\times 1 + 2\times 1 & 3\times 2 + 2\times 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & 8 \\ 7 & 12 \\ 5 & 14 \end{bmatrix}$$

PROBLEM 53. (a) AB is a 2×4 matrix, but BA is not defined.

(b) Neither AB nor BA are defined.
(c) AB is not defined. The matrix BA is a 6×7 matrix.
(d) AB is 2×2 and BA is 3×3.
(e) AB is 3×1, but BA is not defined.
(f) Both A(BC) and (AB)C are 2×4 matrices.
(g) AB is 4×4 and BA is 1×1.

THE ADDITIONAL PROBLEM.

(a) The equivalent matrix equation is:

$$\begin{bmatrix} 2 & 2 & -1 \\ -2 & -2 & 4 \\ 2 & 2 & 5 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ -3 \end{bmatrix}$$

(b) The equivalent vector equation is:

$$x_1 \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 4 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ -3 \end{bmatrix}$$